OPTIMAL RECOVERY OF TRACES IN HARDY SPACES

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Abstract. We construct optimal methods of recovery of traces of analytic functions from the Hardy space $H^2(\mathbb{B}^n)$ on the sphere of radius ρ using inaccurate information about their traces on the spheres of radiuses r_1 and r_2, $r_1 < \rho < r_2$. We show that instead of using the whole information about the trace on the sphere of radius r_1 we can use only its projection on a space of polynomials given with the same accuracy. Moreover, we construct a collection of optimal recovery methods.

1. Introduction

Let $\mathbb{B}^n$ stand for the unit ball in $\mathbb{C}^n$,

$$\mathbb{B}^n = \{ z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z|^2 = \sum_{j=1}^n |z_j|^2 < 1 \}.$$

Recall that the Hardy space $H^2(\mathbb{B}^n)$ consists of all functions f such that

$$\|f\|^2_{H^2(\mathbb{B}^n)} = \sup_{0<r<1} \int_{\mathbb{S}^{n-1}} |f(rz)|^2 \, d\sigma(z) < \infty,$$

where $d\sigma(z)$ is the positive normalized rotationally invariant measure on the unit sphere

$$\mathbb{S}^{n-1} = \{ z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z| = 1 \}.$$

Suppose that for any $f \in H^2(\mathbb{B}^n)$ we know traces of f on the spheres $r_1\mathbb{S}^{n-1}$ and $r_2\mathbb{S}^{n-1}$ given with some accuracy. The problem is to recover the trace of f on the sphere $\rho\mathbb{S}^{n-1}$, $r_1 < \rho < r_2$.

More precisely, suppose that for any $f \in H^2(\mathbb{B}^n)$ we know functions $y_j \in L_2(\sigma_{r_j})$, $j = 1, 2$, where $d\sigma_r(z)$ are the positive normalized rotationally invariant measures on the sphere $r\mathbb{S}^{n-1}$, and

$$\|f(r_j z) - y_j(r_j z)\|_{L_2(\sigma)} \le \delta_j, \quad j = 1, 2.$$

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A recovery algorithm (method, procedure, etc.) is an operator

$$m\colon L^2(\sigma_{r_1})\times L^2(\sigma_{r_2})\to L^2(\sigma_\rho).$$

At this point we impose no conditions on m. In particular, we require m to be neither continuous, nor linear.

Given a recovery method m its accuracy is characterized by the maximal possible error

$$e_\rho(H^2(\mathbb{B}^n),r_1,r_2,\delta_1,\delta_2,m) = \sup_{\substack{f\in H^2(\mathbb{B}^n),\ y_j\in L_2(\sigma_{r_j}),\ j=1,2\\ \|f(r_jz)-y_j(r_jz)\|_{L_2(\sigma)}\le\delta_j,\ j=1,2}} \|f(\rho z)-m(y_1,y_2)(\rho z)\|_{L_2(\sigma)}.$$

We further introduce the optimal recovery error as

$$E_\rho(H^2(\mathbb{B}^n),r_1,r_2,\delta_1,\delta_2) = \inf_{m\colon L^2(\sigma_{r_1})\times L^2(\sigma_{r_2})\to L^2(\sigma_\rho)} e_\rho(H^2(\mathbb{B}^n),r_1,r_2,\delta_1,\delta_2,m). \tag{1}$$

A method $\widehat{m}$ such that

$$E_\rho(H^2(\mathbb{B}^n),r_1,r_2,\delta_1,\delta_2)=e_\rho(H^2(\mathbb{B}^n),r_1,r_2,\delta_1,\delta_2,\widehat{m})$$

is called an *optimal recovery method.*

Let

$$(\widehat{\lambda}_1,\widehat{\lambda}_2)=\left(\frac{r_2^2-\rho^2}{r_2^2-r_1^2}\left(\frac{\rho}{r_1}\right)^{2s},\frac{\rho^2-r_1^2}{r_2^2-r_1^2}\left(\frac{\rho}{r_2}\right)^{2s}\right),$$

if

$$\left(\frac{r_1}{r_2}\right)^{s+1}\le\frac{\delta_1}{\delta_2}<\left(\frac{r_1}{r_2}\right)^{s},\quad s\in\mathbb{Z}_+, \tag{2}$$

and $(\widehat{\lambda}_1,\widehat{\lambda}_2)=(0,1)$, if $\delta_1\ge\delta_2$.

For $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{Z}_+^n$ set $|\alpha|=\alpha_1+\ldots+\alpha_n$, $\alpha!=\alpha_1!\ldots\alpha_n!$, and $z^\alpha=z_1^{\alpha_1}\ldots z_n^{\alpha_n}$.

In [1] we obtained the following result

Theorem 1. *The error of optimal recovery is given by*

$$E_\rho(H^2(\mathbb{B}^n),r_1,r_2,\delta_1,\delta_2)=\sqrt{\widehat{\lambda}_1\delta_1^2+\widehat{\lambda}_2\delta_2^2}$$

and the method

$$\widehat{m}(y_1,y_2)(z)=\sum_{j=0}^\infty\frac{1}{\widehat{\lambda}_1r_1^{2j}+\widehat{\lambda}_2r_2^{2j}}\sum_{|\alpha|=j}(\widehat{\lambda}_1r_1^{2j}c_\alpha^{(1)}+\widehat{\lambda}_2r_2^{2j}c_\alpha^{(2)})z^\alpha,$$

where

$$c_\alpha^{(k)}=\frac{(n+|\alpha|-1)!}{n!\alpha!r_k^{|\alpha|}}\int_{\mathbb{S}^{n-1}}y_k(r_kz)\overline{z}^\alpha\,d\sigma(z),\quad k=1,2,\tag{3}$$

is optimal.

In this paper we construct a collection of optimal recovery methods in the same problem. In particular, we show that there are methods that use only a finite number of coefficients $c_\alpha^{(1)}$ and do not use some first of coefficients $c_\alpha^{(2)}$.

2. Main results

Let I_N be the orthogonal projector of $H^2(\mathbb{B}^n)$ on the space of polynomials of degree N, that is,

$$I_Nf(z)=\sum_{j=0}^N\sum_{|\alpha|=j}c_\alpha z^\alpha,$$

if

$$f(z)=\sum_{j=0}^\infty\sum_{|\alpha|=j}c_\alpha z^\alpha.$$

Let I^N be the orthogonal projector of $H^2(\mathbb{B}^n)$ on the subspace of functions from $H^2(\mathbb{B}^n)$ which are orthogonal to all polynomials of degree N. In other words, $I^Nf=f-I_Nf$. Put $I_\infty f=f$ and $I_{-1}f=0$. Then $I^{-1}f=f$ and $I^\infty f=0$.

We begin with the following recovery problem. To recover the trace f on $\rho\mathbb{S}^{n-1}$ knowing inaccurate traces of I_Nf on $r_1\mathbb{S}^{n-1}$ and I^Mf on $r_2\mathbb{S}^{n-1}$. More precisely, we suppose that for any $f\in H^2(\mathbb{B}^n)$ we know functions $y_j\in L_2(\sigma_{r_j})$, $j=1,2$, where $d\sigma_r(z)$ are the positive normalized rotationally invariant measures on the sphere $r\mathbb{S}^{n-1}$, and

$$\|I_Nf(r_1z)-y_1(r_1z)\|_{L_2(\sigma)}\le\delta_1,\quad \|I^Mf(r_2z)-y_2(r_2z)\|_{L_2(\sigma)}\le\delta_2.\tag{4}$$

Given a recovery method $m\colon L^2(\sigma_{r_1})\times L^2(\sigma_{r_2})\to L^2(\sigma_\rho)$ we define its error as follows

$$\begin{aligned}&e_\rho^{N,M}(H^2(\mathbb{B}^n),r_1,r_2,\delta_1,\delta_2,m)\\&\qquad=\sup_{\substack{f\in H^2(\mathbb{B}^n),\\ y_1\in Y_N(f,r_1,\delta_1),\ y_2\in Y_M(f,r_2,\delta_2)}}\|f(\rho z)-m(y_1,y_2)(\rho z)\|_{L_2(\sigma)},\end{aligned}$$

where

$$\begin{aligned}Y_N(f,r,\delta)&=\{\,y\in L_2(\sigma_r):\|I_Nf(rz)-y(rz)\|_{L_2(\sigma)}\le\delta\,\},\\ Y^M(f,r,\delta)&=\{\,y\in L_2(\sigma_r):\|I^Mf(rz)-y(rz)\|_{L_2(\sigma)}\le\delta\,\}.\end{aligned}$$

We define the optimal recovery error as

$$(5)\quad E_\rho^{N,M}(H^2(\mathbb{B}^n), r_1, r_2, \delta_1, \delta_2) = \inf_{m\colon L^2(\sigma_{r_1})\times L^2(\sigma_{r_2})\to L^2(\sigma_\rho)} e_\rho^{N,M}(H^2(\mathbb{B}^n), r_1, r_2, \delta_1, \delta_2, m).$$

A method $\widehat{m}$ such that

$$E_\rho^{N,M}(H^2(\mathbb{B}^n), r_1, r_2, \delta_1, \delta_2) = e_\rho^{N,M}(H^2(\mathbb{B}^n), r_1, r_2, \delta_1, \delta_2, \widehat{m})$$

is called an *optimal recovery method.*

It is easy to see that for $N = \infty$ and $M = -1$ problem (5) coincides with (1).

The main tool in the solution of optimal recovery problems is the solution of the duality extremal problems. In our case this problem has the form

$$(6)\quad \|f(\rho z)\|^2_{L_2(\sigma)} \to \max, \quad \|I_N f(r_1 z)\|^2_{L_2(\sigma)} \le \delta_1^2, \quad \|I^M f(r_2 z)\|^2_{L_2(\sigma)} \le \delta_2^2, \quad f \in H^2(\mathbb{B}^n).$$

For every method m and for every $f \in H^2(\mathbb{B}^n)$ such that $\|I_N f(r_1 z)\|_{L_2(\sigma)} \le \delta_1$, $\|I^M f(r_2 z)\|_{L_2(\sigma)} \le \delta_2$, we have

$$2\|f(\rho z)\|_{L_2(\sigma)} \le \|f(\rho z) - m(0,0)\|_{L_2(\sigma)} + \|-f(\rho z) - m(0,0)\|_{L_2(\sigma)} \le 2e_\rho^{N,M}(H^2(\mathbb{B}^n), r_1, r_2, \delta_1, \delta_2, m).$$

Hence, for every method m

$$e_\rho^{N,M}(H^2(\mathbb{B}^n), r_1, r_2, \delta_1, \delta_2, m) \ge \sup_{\substack{f\in H^2(\mathbb{B}^n)\\ \|I_N f(r_1 z)\|_{L_2(\sigma)}\le\delta_1\\ \|I^M f(r_2 z)\|_{L_2(\sigma)}\le\delta_2}} \|f(\rho z)\|_{L_2(\sigma)}.$$

Taking the infimum in m, we obtain

$$(7)\quad E_\rho^{N,M}(H^2(\mathbb{B}^n), r_1, r_2, \delta_1, \delta_2) \ge \sup_{\substack{f\in H^2(\mathbb{B}^n)\\ \|I_N f(r_1 z)\|_{L_2(\sigma)}\le\delta_1\\ \|I^M f(r_2 z)\|_{L_2(\sigma)}\le\delta_2}} \|f(\rho z)\|_{L_2(\sigma)}.$$

Note that if $N < M$, then taking $f_0(z) = Az_1^{N+1}$ with $|A| \to \infty$ it is easy to verify that the value in the right-hand side of (7) equals ∞.

Let

$$\mathcal{L}(f, \lambda_1, \lambda_2) = -\|f(\rho z)\|^2_{L_2(\sigma)} + \lambda_1\|I_N f(r_1 z)\|^2_{L_2(\sigma)} + \lambda_2\|I^M f(r_2 z)\|^2_{L_2(\sigma)}$$

be the Lagrange function of problem (6). To construct optimal recovery methods we need the following result.

Theorem 2. *Assume that $N \geq M \geq -1$ and there exist $\widehat{\lambda}_1, \widehat{\lambda}_2 \geq 0$ and $\widehat{f} \in H^2(\mathbb{B}^n)$ admissible in (6) such that*

$$(a) \quad \min_{f \in H^2(\mathbb{B}^n)} \mathcal{L}(f, \widehat{\lambda}_1, \widehat{\lambda}_2) = \mathcal{L}(\widehat{f}, \widehat{\lambda}_1, \widehat{\lambda}_2),$$

$$(b) \quad \widehat{\lambda}_1(\|I_N \widehat{f}(r_1 z)\|^2_{L_2(\sigma)} - \delta_1^2) + \widehat{\lambda}_2(\|I^M \widehat{f}(r_2 z)\|^2_{L_2(\sigma)} - \delta_2^2) = 0.$$

Then the error of optimal recovery is given by

$$E_\rho^{N,M}(H^2(\mathbb{B}^n), r_1, r_2, \delta_1, \delta_2) = \sqrt{\widehat{\lambda}_1 \delta_1^2 + \widehat{\lambda}_2 \delta_2^2}$$

and the method

$$\widehat{m}_{N,M}(y_1, y_2)(z) = \sum_{j=0}^{M} \sum_{|\alpha|=j} c_\alpha^{(1)} z^\alpha$$

$$+ \sum_{j=M+1}^{N} \frac{1}{\widehat{\lambda}_1 r_1^{2j} + \widehat{\lambda}_2 r_2^{2j}} \sum_{|\alpha|=j} (\widehat{\lambda}_1 r_1^{2j} c_\alpha^{(1)} + \widehat{\lambda}_2 r_2^{2j} c_\alpha^{(2)}) z^\alpha + \sum_{j=N+1}^{\infty} \sum_{|\alpha|=j} c_\alpha^{(2)} z^\alpha,$$

where $c_\alpha^{(k)}$, $k = 1, 2$, are defined by (3), is optimal.

Proof. Set

$$S^2 = \widehat{\lambda}_1 \delta_1^2 + \widehat{\lambda}_2 \delta^2.$$

Let $f \in H^2(\mathbb{B}^n)$ be an admissible element in (6). Then

$$\begin{aligned} -\|f(\rho z)\|^2_{L_2(\sigma)} &\geq -\|f(\rho z)\|^2_{L_2(\sigma)} + \widehat{\lambda}_1(\|I_N f(r_1 z)\|^2_{L_2(\sigma)} - \delta_1^2) \\ &+ \widehat{\lambda}_2(\|I^M f(r_2 z)\|^2_{L_2(\sigma)} - \delta_2^2) = \mathcal{L}(f, \widehat{\lambda}_1, \widehat{\lambda}_2) - S^2 \geq \mathcal{L}(\widehat{f}, \widehat{\lambda}_1, \widehat{\lambda}_2) - S^2 \\ &= -\|\widehat{f}(\rho z)\|^2_{L_2(\sigma)} + \widehat{\lambda}_1(\|I_N \widehat{f}(r_1 z)\|^2_{L_2(\sigma)} - \delta_1^2) + \widehat{\lambda}_2(\|I^M \widehat{f}(r_2 z)\|^2_{L_2(\sigma)} - \delta_2^2) \\ &= -\|\widehat{f}(\rho z)\|^2_{L_2(\sigma)}. \end{aligned}$$

Hence, $\widehat{f}$ is a solution of (6). The same arguments show that $\widehat{f}$ is a solution of the extremal problem

$$(8) \quad \|f(\rho z)\|^2_{L_2(\sigma)} \to \max, \quad \widehat{\lambda}_1 \|I_N f(r_1 z)\|^2_{L_2(\sigma)} + \widehat{\lambda}_2 \|I^M f(r_2 z)\|^2_{L_2(\sigma)} \leq \widehat{\lambda}_1 \delta_1^2 + \widehat{\lambda}_2 \delta_2^2, \quad f \in H^2(\mathbb{B}^n).$$

Since for every $c \in \mathbb{C}$, $\mathcal{L}(cf, \widehat{\lambda}_1, \widehat{\lambda}_2) = |c|^2 \mathcal{L}(f, \widehat{\lambda}_1, \widehat{\lambda}_2)$ we have $\mathcal{L}(\widehat{f}, \widehat{\lambda}_1, \widehat{\lambda}_2) = 0$. Consequently,

$$\|\widehat{f}(\rho z)\|^2_{L_2(\sigma)} = -\mathcal{L}(\widehat{f}, \widehat{\lambda}_1, \widehat{\lambda}_2) + S^2 = S^2.$$

Thus we proved that

$$\sup_{\substack{f\in H^2(\mathbb{B}^n)\\ \|I_N f(r_1z)\|_{L_2(\sigma)}\le\delta_1\\ \|I^M f(r_2z)\|_{L_2(\sigma)}\le\delta_2}} \|f(\rho z)\|_{L_2(\sigma)}$$

$$= \sup_{\substack{f\in H^2(\mathbb{B}^n)\\ \widehat{\lambda}_1\|I_N f(r_1z)\|^2_{L_2(\sigma)}+\widehat{\lambda}_2\|I^M f(r_2z)\|^2_{L_2(\sigma)}\le\widehat{\lambda}_1\delta_1^2+\widehat{\lambda}_2\delta_2^2}} \|f(\rho z)\|_{L_2(\sigma)} = S,$$

and moreover,

$$E_\rho^{N,M}(H^2(\mathbb{B}^n), r_1, r_2, \delta_1, \delta_2) \ge S.$$

Now we obtain the lower bound. For any functions $y_k \in L_2(\sigma_{r_k})$, $k = 1, 2$, consider the extremal problem

$$\begin{aligned}(9)\quad \widehat{\lambda}_1\|I_N f(r_1z) - y_1(r_1z)\|^2_{L_2(\sigma)} \\ + \widehat{\lambda}_2\|I^M f(r_2z) - y_2(r_2z)\|^2_{L_2(\sigma)} \to \min, \quad f \in H^2(\mathbb{B}^n).\end{aligned}$$

Write $y_k(r_kz)$, $k = 1, 2$,

$$y_k(r_kz) = \sum_{j=0}^{\infty} r_k^j \sum_{\substack{|\alpha|=j\\ \alpha\ge 0}} c_\alpha^{(k)} z^\alpha + \widetilde{y}_k(z),$$

where $\widetilde{y}_k$, $k = 1, 2$, are orthogonal to all holomorphic polynomials in $L_2(\sigma)$. Then problem (9) may be written in the form

$$\begin{aligned}&\widehat{\lambda}_1 \sum_{j=0}^{N} \frac{r_1^{2j} n!}{(n+j-1)!} \sum_{|\alpha|=j} \alpha!|f_\alpha - c_\alpha^{(1)}|^2 + \|y_1^*\|^2_{L_2(\sigma)} \\ &\quad + \widehat{\lambda}_2 \sum_{j=M+1}^{\infty} \frac{r_2^{2j} n!}{(n+j-1)!} \sum_{|\alpha|=j} \alpha!|f_\alpha - c_\alpha^{(2)}|^2 + \|y_2^*\|^2_{L_2(\sigma)} \to \min, \\ &\qquad\qquad\qquad\qquad f \in H^2(\mathbf{B}_n),\end{aligned}$$

where

$$y_1^*(z) = \sum_{j=N+1}^{\infty} r_1^j \sum_{|\alpha|=j} c_\alpha^{(1)} z^\alpha + \widetilde{y}_1(z), \quad y_2^*(z) = \sum_{j=0}^{M} r_2^j \sum_{|\alpha|=j} c_\alpha^{(2)} z^\alpha + \widetilde{y}_2(z).$$

It is easy to show that for all functions $y_1(r_1z), y_2(r_2z) \in L_2(\sigma)$ with finite number of coefficients $c_\alpha^{(k)} \neq 0$, $k = 1, 2$ (we denote this space of functions by $\mathcal{P}$), the solution of this problem is

$$f_y(z) = \sum_{j=0}^{M} \sum_{|\alpha|=j} c_\alpha^{(1)} z^\alpha$$
$$+ \sum_{j=M+1}^{N} \frac{1}{\widehat{\lambda}_1 r_1^{2j} + \widehat{\lambda}_2 r_2^{2j}} \sum_{|\alpha|=j} (\widehat{\lambda}_1 r_1^{2j} c_\alpha^{(1)} + \widehat{\lambda}_2 r_2^{2j} c_\alpha^{(2)}) z^\alpha + \sum_{j=N+1}^{\infty} \sum_{|\alpha|=j} c_\alpha^{(2)} z^\alpha,$$

where $y = (y_1, y_2)$. Now consider the linear space $E = L^2(\sigma_{r_1}) \times L^2(\sigma_{r_2})$ with the semi-inner product

$$(y^1, y^2)_E = \widehat{\lambda}_1 (y_1^1, y_1^2)_{L^2(\sigma_{r_1})} + \widehat{\lambda}_2 (y_2^1, y_2^2)_{L^2(\sigma_{r_2})},$$

where $y^1 = (y_1^1, y_2^1)$, $y^2 = (y_1^2, y_2^2)$. Then (9) can be written in the form

$$\|\widetilde{f} - y\|_E^2 \to \min, \quad f \in H^2(\mathbb{B}^n), \tag{10}$$

where $\widetilde{f} = (I_N f, I^M f)$. If f_y is a solution of (10) it can be easily verified that for all $f \in H^2(\mathbb{B}^n)$

$$(\widetilde{f}_y - y, \widetilde{f})_E = 0.$$

Consequently,

$$\|\widetilde{f} - y\|_E^2 = \|\widetilde{f} - \widetilde{f}_y\|_E^2 + \|\widetilde{f}_y - y\|_E^2.$$

Thus, for all $f \in H^2(\mathbb{B}^n)$

$$\begin{aligned} \|\widetilde{f} - \widetilde{f}_y\|_E^2 &\le \|\widetilde{f} - y\|_E^2 \\ &= \widehat{\lambda}_1 \|I_N f(r_1 z) - y_1(r_1 z)\|_{L_2(\sigma)}^2 + \widehat{\lambda}_2 \|I^M f(r_2 z) - y_2(r_2 z)\|_{L_2(\sigma)}^2. \end{aligned} \tag{11}$$

Let $f \in H^2(\mathbb{B}^n)$ and $y \in E$ satisfies (4). Then for any $\varepsilon > 0$ there exists $y_1^*(r_1 z), y_2^*(r_2 z) \in \mathcal{P}$ such that $\|y_k(r_k z) - y_k^*(r_k z)\|_{L_2(\sigma)} < \varepsilon$, $k = 1, 2$. Thus,

$$\begin{aligned} \|I_N f(r_1 z) - y_1^*(r_1 z)\|_{L_2(\sigma)} &\le \|I_N f(r_1 z) - y_1(r_1 z)\|_{L_2(\sigma)} \\ &\quad + \|y_1(r_1 z) - y_1^*(r_1 z)\|_{L_2(\sigma)} \le \delta_1 + \varepsilon \end{aligned}$$

and

$$\begin{aligned} \|I^M f(r_2 z) - y_2^*(r_2 z)\|_{L_2(\sigma)} &\le \|I^M f(r_2 z) - y_2(r_2 z)\|_{L_2(\sigma)} \\ &\quad + \|y_2(r_2 z) - y_2^*(r_2 z)\|_{L_2(\sigma)} \le \delta_2 + \varepsilon. \end{aligned}$$

Set $g = f - f_{y^*}$ where $y^* = (y_1^*, y_2^*)$. Then (11) implies that

$$\begin{aligned} \widehat{\lambda}_1 \|I_N g(r_1 z)\|_{L_2(\sigma)}^2 + \widehat{\lambda}_2 \|I^M g(r_2 z)\|_{L_2(\sigma)}^2 &= \|\widetilde{g}\|_E^2 \\ &\le \widehat{\lambda}_1 (\delta_1 + \varepsilon)^2 + \widehat{\lambda}_2 (\delta_2 + \varepsilon)^2. \end{aligned}$$

We have the following estimate for the method $\widehat{m}$

$$\|f(\rho z)-\widehat{m}(y_1,y_2)(\rho z)\|_{L_2(\sigma)} \le \|f(\rho z)-\widehat{m}(y_1^*,y_2^*)(\rho z)\|_{L_2(\sigma)} \\ + \|\widehat{m}(y_1-y_1^*,y_2-y_2^*)(\rho z)\|_{L_2(\sigma)} \le \|g(\rho z)\|_{L_2(\sigma)} + C\varepsilon$$

(the value of constant C depends on $\widehat{\lambda}_1$, $\widehat{\lambda}_2$, r_1, r_2, and ρ but is not significant for us).

Taking into account that for all $C_1, C_2 > 0$

$$\sup_{\substack{f\in H^2(\mathbb{B}^n)\\ \widehat{\lambda}_1\|I_N f(r_1z)\|^2_{L_2(\sigma)}+\widehat{\lambda}_2\|I^M f(r_2z)\|^2_{L_2(\sigma)}\le C_1}} \|f(\rho z)\|_{L_2(\sigma)} \\ = \frac{C_1}{C_2} \sup_{\substack{f\in H^2(\mathbb{B}^n)\\ \widehat{\lambda}_1\|I_N f(r_1z)\|^2_{L_2(\sigma)}+\widehat{\lambda}_2\|I^M f(r_2z)\|^2_{L_2(\sigma)}\le C_2}} \|f(\rho z)\|_{L_2(\sigma)},$$

we obtain

$$\|f(\rho z)-\widehat{m}(y_1^*,y_2^*)(\rho z)\|^2_{L_2(\sigma)} = \|g(\rho z)\|^2_{L_2(\sigma)} \\ \le \sup_{\substack{f\in H^2(\mathbb{B}^n)\\ \widehat{\lambda}_1\|I_N f(r_1z)\|^2_{L_2(\sigma)}+\widehat{\lambda}_2\|I^M f(r_2z)\|^2_{L_2(\sigma)}\le\widehat{\lambda}_1(\delta_1+\varepsilon)^2+\widehat{\lambda}_2(\delta_2+\varepsilon)^2}} \|f(\rho z)\|^2_{L_2(\sigma)} \\ = \frac{S^2+\varepsilon^2(\widehat{\lambda}_1+\widehat{\lambda}_2)}{S^2} \sup_{\substack{f\in H^2(\mathbb{B}^n)\\ \widehat{\lambda}_1\|I_N f(r_1z)\|^2_{L_2(\sigma)}+\widehat{\lambda}_2\|I^M f(r_2z)\|^2_{L_2(\sigma)}\le S^2}} \|f(\rho z)\|^2_{L_2(\sigma)} \\ = S^2+\varepsilon^2(\widehat{\lambda}_1+\widehat{\lambda}_2).$$

Since $\varepsilon > 0$ is arbitrary we obtain

$$E_\rho^{N,M}(H^2(\mathbb{B}^n), r_1, r_2, \delta_1, \delta_2) \le e_\rho^{N,M}(H^2(\mathbb{B}^n), r_1, r_2, \delta_1, \delta_2, \widehat{m}) \le S.$$

This and (7) imply

$$E_\rho^{N,M}(H^2(\mathbb{B}^n), r_1, r_2, \delta_1, \delta_2) = S$$

and $\widehat{m}$ is an optimal method. □

First we find the error of optimal recovery. We introduce the following notation:

$$s_0 = (N+\lceil\alpha\rceil)_+, \quad \alpha = 1 - \frac{\log\dfrac{r_2^2-r_1^2}{\rho^2-r_1^2}}{2\log\dfrac{r_2}{\rho}}, \tag{12}$$

where $\lceil x \rceil = \inf\{\, n \in \mathbb{Z} : n \geq x \,\}$ is the ceiling function and $(x)_+ = \max\{x, 0\}$, $a(-1) = -1$, for $N \geq 0$

$$a(N) = \left(s_0 + \log \left(1 - \left(\frac{\rho}{r_2} \right)^{2(N-s_0+1)} \right) \Big/ \left(2 \log \frac{\rho}{r_1} \right) \right)_+ ,$$

$$s_1 = \left\lceil \frac{\log(\delta_2/\delta_1)}{\log(r_2/r_1)} \right\rceil , \quad \widehat{N} = \begin{cases} -\lceil \alpha \rceil + s_1, & \delta_2 > \delta_1, \\ 0, & \delta_2 \leq \delta_1, \end{cases}$$

$$\widehat{M} = \left(s_1 - \log \frac{r_2^2 - r_1^2}{r_2^2 - \rho^2} \Big/ \left(2 \log \frac{\rho}{r_1} \right) \right)_+ - 1.$$

For $N \geq M \geq -1$ consider the sets:

$$\Sigma_1 = \{\, (N, M) : a(N) \leq M \leq N \,,\ M \geq 0 \} \,,$$

$$\Sigma_2 = \left\{ (N, M) : -1 \leq M < a(N),\ N < \widehat{N} \right\} \cup (-1, -1),$$

$$\Sigma_3 = \left\{ (N, M) : -1 \leq M \leq \widehat{M},\ N \geq \widehat{N} \right\},$$

$$\Sigma_4 = \left\{ (N, M) : \widehat{M} < M < a(N),\ N \geq \widehat{N} \right\}.$$

Theorem 3. *The error of optimal recovery is given by*

$$E_\rho^{N,M}(H^2(\mathbb{B}^n), r_1, r_2, \delta_1, \delta_2) = \sqrt{\widehat{\lambda}_1 \delta_1^2 + \widehat{\lambda}_2 \delta_2^2},$$

where

$$\widehat{\lambda}_1 = \begin{cases} \left(\frac{\rho}{r_1} \right)^{2M}, & (N, M) \in \Sigma_1 \cup \Sigma_4, \\ \left(\frac{\rho}{r_1} \right)^{2s_0} \left(1 - \left(\frac{\rho}{r_2} \right)^{2(N-s_0+1)} \right), & (N, M) \in \Sigma_2, \\ \left(\frac{\rho}{r_1} \right)^{2(s_1-1)} \frac{r_2^2 - \rho^2}{r_2^2 - r_1^2}, & (N, M) \in \Sigma_3,\ \delta_2 > \delta_1, \\ 0, & (N, M) \in \Sigma_3,\ \delta_2 \leq \delta_1, \end{cases}$$

$$\widehat{\lambda}_2 = \begin{cases} \left(\frac{\rho}{r_2} \right)^{2(N+1)}, & (N, M) \in \Sigma_1 \cup \Sigma_2, \\ \left(\frac{\rho}{r_2} \right)^{2(s_1-1)} \frac{\rho^2 - r_1^2}{r_2^2 - r_1^2}, & (N, M) \in \Sigma_3,\ \delta_2 > \delta_1, \\ 1, & (N, M) \in \Sigma_3,\ \delta_2 \leq \delta_1, \\ \left(\frac{\rho}{r_2} \right)^{2s_2} \left(1 - \left(\frac{\rho}{r_1} \right)^{2(M-s_2)} \right), & (N, M) \in \Sigma_4, \end{cases}$$

and

$$s_2 = M + \left\lceil \frac{\log \dfrac{r_2^2 - r_1^2}{r_2^2 - \rho^2}}{2 \log \dfrac{\rho}{r_1}} \right\rceil . \tag{13}$$

Proof. In accordance with Theorem 2 we should find $\widehat{\lambda}_1$, $\widehat{\lambda}_2$, and $\widehat{f} \in H^2(\mathbb{B}^n)$ admissible in (6) which satisfy conditions (a) and (b). Assume that $f \in H^2(\mathbb{B}^n)$, then it can be represented in the form

$$f(z) = \sum_{j=0}^{\infty} \sum_{|\alpha|=j} c_\alpha z^\alpha .$$

Since monomials z^α form an orthogonal system in $H^2(\mathbf{B}_n)$ with

$$\|z^\alpha\|^2_{H^2(\mathbf{B}_n)} = \frac{n!\alpha!}{(n+|\alpha|-1)!},$$

(see [2], sect. 1.4.9) we have

$$\begin{aligned}\mathcal{L}(f, \lambda_1, \lambda_2) = \sum_{j=0}^{M} r_1^{2j} \left(-\left(\frac{\rho}{r_1}\right)^{2j} + \lambda_1 \right) b_j \\ + \sum_{j=M+1}^{N} r_1^{2j} \left(-\left(\frac{\rho}{r_1}\right)^{2j} + \lambda_1 + \lambda_2 \left(\frac{r_2}{r_1}\right)^{2j} \right) b_j \\ + \sum_{j=N+1}^{\infty} r_1^{2j} \left(-\left(\frac{\rho}{r_1}\right)^{2j} + \lambda_2 \left(\frac{r_2}{r_1}\right)^{2j} \right) b_j,\end{aligned}$$

where

$$b_j = \frac{n!}{(n+j-1)!} \sum_{|\alpha|=j} \alpha! |c_\alpha|^2 .$$

Consider the set of points on the plane $\mathbb{R}^2$

$$\begin{cases} y_j = \left(\dfrac{\rho}{r_1}\right)^{2j}, \\ x_j = \left(\dfrac{r_2}{r_1}\right)^{2j}, \end{cases} \quad j = 0, 1, \dots . \tag{14}$$

These points belong to the plot of concave function

$$\begin{cases} y = \left(\dfrac{\rho}{r_1}\right)^{2t}, \\ x = \left(\dfrac{r_2}{r_1}\right)^{2t}, \end{cases} \quad t \in [0, \infty).$$

Consequently, the piecewise linear function passing through the points (14) is also concave. There exists the minimal integer number s_0, $0 \le s_0 \le N$, such that all points (14) lie under the line passing through the point (x_{s_0}, y_{s_0}) with the slope equals y_{N+1}/x_{N+1}. The number s_0 may be formally defined as follows

$$s_0 = \min\left\{ s \in \mathbb{Z}_+ : \frac{y_{s+1} - y_s}{x_{s+1} - x_s} \le \frac{y_{N+1}}{x_{N+1}} \right\}.$$

It can be easily verified that this number may be also defined by (12).

Let $y = \widehat{\lambda}_1 + \widehat{\lambda}_2 x$ be the line

$$y - y_{s_0} = \frac{y_{N+1}}{x_{N+1}}(x - x_{s_0}),$$

that is,

$$\widehat{\lambda}_1 = \left(\frac{\rho}{r_1}\right)^{2s_0}\left(1 - \left(\frac{\rho}{r_2}\right)^{2(N-s_0+1)}\right), \quad \widehat{\lambda}_2 = \left(\frac{\rho}{r_2}\right)^{2(N+1)}. \tag{15}$$

Then for all $j \in \mathbb{Z}_+$

$$-\left(\frac{\rho}{r_1}\right)^{2j} + \widehat{\lambda}_1 + \widehat{\lambda}_2\left(\frac{r_2}{r_1}\right)^{2j} \ge 0.$$

Assume that $M \ge 0$ and

$$\left(\frac{\rho}{r_1}\right)^{2M} \ge \widehat{\lambda}_1.$$

It means that

$$M \ge \left(s_0 + \log\left(1 - \left(\frac{\rho}{r_2}\right)^{2(N-s_0+1)}\right) \Big/ \left(2\log\frac{\rho}{r_1}\right)\right)_+ = a(N).$$

Then putting

$$\widehat{\lambda}_1' = \left(\frac{\rho}{r_1}\right)^{2M}$$

we have

$$\mathcal{L}(f, \widehat{\lambda}_1', \widehat{\lambda}_2) \ge 0$$

for all $f \in H^2(\mathbb{B}^n)$. Consider the function

$$\widehat{f}(z) = \frac{\delta_1}{r_1^M}\sqrt{\frac{(n+M-1)!}{n!M!}}z_1^M + \frac{\delta_2}{r_2^{N+1}}\sqrt{\frac{(n+N)!}{n!(N+1)!}}z_1^{N+1}.$$

It is easy to verify that $\widehat{f} \in H^2(\mathbb{B}^n)$ is admissible in (6) and satisfies (b). Moreover, $\mathcal{L}(\widehat{f}, \widehat{\lambda}_1', \widehat{\lambda}_2) = 0$. Hence condition (a) is also fulfilled. Applying Theorem 2 we obtain that for $(N, M) \in \Sigma_1$

$$E_\rho^{N,M}(H^2(\mathbb{B}^n), r_1, r_2, \delta_1, \delta_2) = \sqrt{\left(\frac{\rho}{r_1}\right)^{2M} \delta_1^2 + \left(\frac{\rho}{r_2}\right)^{2(N+1)} \delta_2^2}.$$

Assume now that $M = N = -1$. Then for all $f \in H^2(\mathbb{B}^n)$, $\mathcal{L}(f, 0, 1) \geq 0$. For $\widehat{f}(z) = \delta_2$

$$\|I_{-1}\widehat{f}(r_1 z)\|^2_{L_2(\sigma)} = 0, \quad \|I^{-1}\widehat{f}(r_2 z)\|^2_{L_2(\sigma)} = \delta_2^2.$$

Thus from Theorem 2 we obtain that $E_\rho^{-1,-1}(H^2(\mathbb{B}^n), r_1, r_2, \delta_1, \delta_2) = \delta_2$.

If $\delta_2 \leq \delta_1$, then $\widehat{N} = 0$ and $\Sigma_2 = (-1, 1)$. Suppose that $\delta_2 > \delta_1$. It is easy to show that in this case

$$\widehat{N} = \min\left\{N \in \mathbb{N} : \frac{\delta_2}{\delta_1} \geq \left(\frac{r_2}{r_1}\right)^{s_0}\right\}.$$

Assume that $-1 \leq M < a(N)$ and $N < \widehat{N}$. Then

$$\left(\frac{\rho}{r_1}\right)^{2M} < \widehat{\lambda}_1$$

and for all $f \in H^2(\mathbb{B}^n)$, $\mathcal{L}(f, \widehat{\lambda}_1, \widehat{\lambda}_2) \geq 0$. Consider the function

$$\widehat{f}(z) = \frac{\delta_1}{r_1^{s_0}} \sqrt{\frac{(n + s_0 - 1)!}{n! s_0!}} z_1^{s_0} + \frac{a}{r_2^{N+1}} \sqrt{\frac{(n+N)!}{n!(N+1)!}} z_1^{N+1},$$

where a will be defined later. We have

$$\|I_N \widehat{f}(r_1 z)\|^2_{L_2(\sigma)} = \delta_1^2, \quad \|I^M \widehat{f}(r_2 z)\|^2_{L_2(\sigma)} = \left(\frac{r_2}{r_1}\right)^{2s_0} \delta_1^2 + a^2.$$

Since $N < \widehat{N}$

$$\delta_2^2 - \left(\frac{r_2}{r_1}\right)^{2s_0} \delta_1^2 > 0$$

and we can choose the parameter a so that $\widehat{f}$ is admissible in (6) and satisfies (b). Hence using Theorem 2 the case $(M, N) \in \Sigma_2$ is proved.

Now assume that

$$\frac{\delta_2}{\delta_1} \leq \left(\frac{r_2}{r_1}\right)^{s_0}.$$

It means that $N \geq \widehat{N}$. Let us find $s_1 \leq s_0$ from the condition

$$\left(\frac{r_2}{r_1}\right)^{s_1 - 1} < \frac{\delta_2}{\delta_1} \leq \left(\frac{r_2}{r_1}\right)^{s_1}. \tag{16}$$

Hence

$$s_1 = \left\lceil \frac{\log(\delta_2/\delta_1)}{\log(r_2/r_1)} \right\rceil .$$

Suppose that $s_1 \geq 1$. Consider the line, which passes through the points (x_{s_1-1}, y_{s_1-1}) and (x_{s_1}, y_{s_1}). It has the form $y = \widehat{\lambda}_1 + \widehat{\lambda}_2 x$, where now

$$\widehat{\lambda}_1 = \left(\frac{\rho}{r_1}\right)^{2(s_1-1)} \frac{r_2^2 - \rho^2}{r_2^2 - r_1^2}, \quad \widehat{\lambda}_2 = \left(\frac{\rho}{r_2}\right)^{2(s_1-1)} \frac{\rho^2 - r_1^2}{r_2^2 - r_1^2}. \tag{17}$$

In view of concavity of the piecewise linear function passing through the points (14) for all these points

$$-y_j + \widehat{\lambda}_1 + \widehat{\lambda}_2 x_j \geq 0.$$

Since the values

$$\left(\frac{\rho}{r_2}\right)^{2(s-1)} \frac{\rho^2 - r_1^2}{r_2^2 - r_1^2}$$

decrease as $s \to \infty$ we have

$$\widehat{\lambda}_2 \geq \left(\frac{\rho}{r_2}\right)^{2(s_0-1)} \frac{\rho^2 - r_1^2}{r_2^2 - r_1^2} \geq \left(\frac{\rho}{r_2}\right)^{2(N+1)}.$$

Thus if $M = -1$ or $M \geq 0$ and

$$\widehat{\lambda}_1 \geq \left(\frac{\rho}{r_1}\right)^{2M}, \tag{18}$$

then for all $f \in H^2(\mathbb{B}^n)$, $\mathcal{L}(f, \widehat{\lambda}_1, \widehat{\lambda}_2) \geq 0$. Condition (18) can be rewritten in the form

$$M \leq s_1 - 1 - \frac{\log \dfrac{r_2^2 - r_1^2}{r_2^2 - \rho^2}}{2 \log \dfrac{\rho}{r_1}}.$$

Consider the function

$$\widehat{f}(z) = a\sqrt{\frac{(n + s_1 - 2)!}{n!(s_1 - 1)!}} z_1^{s_1-1} + b\sqrt{\frac{(n + s_1 - 1)!}{n! s_1!}} z_1^{s_1}.$$

We want to find the parameters a and b from the conditions

$$\|I_N \widehat{f}(r_1 z)\|^2_{L_2(\sigma)} = \delta_1^2, \quad \|I^M \widehat{f}(r_2 z)\|^2_{L_2(\sigma)} = \delta_2^2. \tag{19}$$

It means that

$$a^2 r_1^{2(s_1-1)} + b^2 r_1^{2s_1} = \delta_1^2,$$
$$a^2 r_2^{2(s_1-1)} + b^2 r_2^{2s_1} = \delta_1^2.$$

It is easy to verify that condition (16) allows to find such parameters. Thus we constructed the function $\widehat{f}$ which is admissible in (6) and satisfies (b). From Theorem 2 we obtain the value of the optimal recovery error.

Now suppose that $s_1 < 1$. It means that $\delta_2 \le \delta_1$. In this case $\widehat{M} = -1$. Then for all $f \in H^2(\mathbb{B}^n)$, $\mathcal{L}(f, 0, 1) \ge 0$. For $\widehat{f}(z) = \delta_2$

$$\|I_N\widehat{f}(r_1 z)\|^2_{L_2(\sigma)} = \delta_2^2 \le \delta_1^2, \quad \|I^{-1}\widehat{f}(r_2 z)\|^2_{L_2(\sigma)} = \delta_2^2.$$

Thus from Theorem 2 we obtain that $E_\rho^{N,-1}(H^2(\mathbb{B}^n), r_1, r_2, \delta_1, \delta_2) = \delta_2$.

Finally, consider the case when $\widehat{M} < M < a(N)$ and $N \ge \widehat{N}$ (that is, $(N, M) \in \Sigma_4$). Let $y = \widehat{\lambda}_1 + \widehat{\lambda}_2 x$ be the tangent to the piecewise linear function passing through the points (14) that passes through the point $(0, (\rho/r_1)^{2M})$. That is

$$\widehat{\lambda}_1 = \left(\frac{\rho}{r_1}\right)^{2M}, \quad \widehat{\lambda}_2 = \left(\frac{\rho}{r_2}\right)^{2s_2}\left(1 - \left(\frac{\rho}{r_1}\right)^{2(M-s_2)}\right),$$

where s_2 be the minimal number such that the tangent contains the point (x_{s_2}, y_{s_2}) from the set of points (14). Thus, $s_1 \le s_2 \le s_0$. The number s_2 may be defined as the number satisfying the condition

$$\frac{y_{s_2} - y_{s_2-1}}{x_{s_2} - x_{s_2-1}} > \frac{y_{s_2} - \left(\dfrac{\rho}{r_1}\right)^{2M}}{x_{s_2}} \ge \frac{y_{s_2+1} - y_{s_2}}{x_{s_2+1} - x_{s_2}}.$$

It can be shown that s_2 may be defined by (13). Again for all points (14)

$$-y_j + \widehat{\lambda}_1 + \widehat{\lambda}_2 x_j \ge 0$$

and for all $f \in H^2(\mathbb{B}^n)$, $\mathcal{L}(f, \widehat{\lambda}_1, \widehat{\lambda}_2) \ge 0$. Consider the function

$$\widehat{f}(z) = a\sqrt{\frac{(n+M-1)!}{n!M!}}z_1^M + b\sqrt{\frac{(n+s_2-1)!}{n!s_2!}}z_1^{s_2}.$$

Now to satisfy conditions (19) we have to choose a and b such that

$$\begin{aligned} a^2 r_1^{2M} + b^2 r_1^{2s_2} &= \delta_1^2, \\ b^2 r_2^{2s_2} &= \delta_2^2. \end{aligned}$$

Taking into account (16) we have

$$\left(\frac{r_1}{r_2}\right)^{s_2} \le \left(\frac{r_1}{r_2}\right)^{s_1} \le \frac{\delta_1}{\delta_2}.$$

Hence we can find parameters a and b to satisfy (19). It remains to use Theorem 2. $\square$

Now we construct a family of optimal recovery methods for the sets Σ_j, $j = 2, 3, 4$. Put

$$(N', M') = \begin{cases} (N, M), & (N, M) \in \Sigma_1, \\ (N, a(N)), & (N, M) \in \Sigma_2, \\ (\widehat{N}, \widehat{M}), & (N, M) \in \Sigma_3, \\ (a^{-1}(M), M), & (N, M) \in \Sigma_4, \end{cases}$$

where

$$a^{-1}(M) = \min\{\, N \in \mathbb{Z}_+ : a(N) \geq M \,\}.$$

Theorem 4. *For all integer $\widetilde{N}$ and $\widetilde{M}$ such that $N' \leq \widetilde{N} \leq N$ and $M \leq \widetilde{M} \leq M'$ the methods $\widehat{m}_{\widetilde{N},\widetilde{M}}(y_1, y_2)$ are optimal for problem* (5).

Proof. Let $f \in H^2(\mathbb{B}^n)$, $y_1 \in Y_N(f, r_1, \delta_1)$, $y_2 \in Y^M(f, r_2, \delta_2)$, and

$$y_k(r_k z) = \sum_{j=0}^{\infty} r_k^j \sum_{\substack{|\alpha|=j \\ \alpha \geq 0}} c_\alpha^{(k)} z^\alpha + \widetilde{y}_k(z), \quad k = 1, 2,$$

where $\widetilde{y}_k$, $k = 1, 2$, are orthogonal to all holomorphic polynomials in $L_2(\sigma)$. Then

$$\|f(\rho z) - \widehat{m}_{\widetilde{N},\widetilde{M}}(y_1, y_2)(\rho z)\|_{L_2(\sigma)} = \|f(\rho z) - \widehat{m}_{\widetilde{N},\widetilde{M}}(y_1', y_2')(\rho z)\|_{L_2(\sigma)},$$

where

$$y_1'(z) = \sum_{j=0}^{\widetilde{N}} \sum_{\substack{|\alpha|=j \\ \alpha \geq 0}} c_\alpha^{(1)} z^\alpha, \quad y_2'(z) = \sum_{\widetilde{M}+1}^{\infty} \sum_{\substack{|\alpha|=j \\ \alpha \geq 0}} c_\alpha^{(2)} z^\alpha.$$

Moreover, $y_1' \in Y_{\widetilde{N}}(f, r_1, \delta_1)$, $y_2' \in Y^{\widetilde{M}}(f, r_2, \delta_2)$. Thus

$$e_\rho^{N,M}(H^2(\mathbb{B}^n), r_1, r_2, \delta_1, \delta_2, \widehat{m}_{\widetilde{N},\widetilde{M}}) \leq e_\rho^{\widetilde{N},\widetilde{M}}(H^2(\mathbb{B}^n), r_1, r_2, \delta_1, \delta_2, \widehat{m}_{\widetilde{N},\widetilde{M}}).$$

From this inequality we have

$$\begin{aligned} E_\rho^{N,M}(H^2(\mathbb{B}^n), r_1, r_2, \delta_1, \delta_2) &\leq e_\rho^{N,M}(H^2(\mathbb{B}^n), r_1, r_2, \delta_1, \delta_2, \widehat{m}_{\widetilde{N},\widetilde{M}}) \\ &\leq e_\rho^{\widetilde{N},\widetilde{M}}(H^2(\mathbb{B}^n), r_1, r_2, \delta_1, \delta_2, \widehat{m}_{\widetilde{N},\widetilde{M}}) = E_\rho^{\widetilde{N},\widetilde{M}}(H^2(\mathbb{B}^n), r_1, r_2, \delta_1, \delta_2). \end{aligned}$$

Using the fact that

$$E_\rho^{N,M}(H^2(\mathbb{B}^n), r_1, r_2, \delta_1, \delta_2) = E_\rho^{\widetilde{N},\widetilde{M}}(H^2(\mathbb{B}^n), r_1, r_2, \delta_1, \delta_2)$$

we obtain that the methods $\widehat{m}_{\widetilde{N},\widetilde{M}}$ are optimal. □

Returning to problem (1) we obtain

Corollary 1. *For all integer N and M such that $\widehat{N} \leq N < \infty$ and $-1 \leq M \leq \widehat{M}$ the methods $\widehat{m}_{N,M}(y_1, y_2)$ are optimal for problem* (1).

Now we would like to consider the following question: is it possible to choose a best method in some sense among all these optimal methods. Usually numerical algorithms (for instance, interpolation or quadrature formulae) considered as good ones if they are precise for some set of functions (for instance, subspaces of algebraic or trigonometric polynomials). In this connection one tries to make this set as large as possible. For example, Gauss quadrature is constructed to make the largest value of n so that all polynomials of degree at most n are integrated precisely.

We say that a method $m(y_1, y_2)$ is precise by first (second) argument at a function $f \in H^2(\mathbb{B}^n)$ for problem (5) if

$$m(I_N f, 0) = f \quad (m(0, I^M f) = f).$$

We say that a method $m(y_1, y_2)$ is precise by first (second) argument at a set W if it is precise by first (second) argument at all functions from W.

It is easily seen that for all $N' \le \widetilde{N} \le N$ and $M \le \widetilde{M} \le M'$ the methods $\widehat{m}_{\widetilde{N},\widetilde{M}}(y_1, y_2)$ are precise by first (second) argument at the spaces $\mathcal{P}_{\widetilde{M}}$ ($\mathcal{P}^{\perp}_{\widetilde{N}}$), where $\mathcal{P}_{\widetilde{M}}$ is the space of polynomials of degree at most $\widetilde{M}$ and $\mathcal{P}^{\perp}_{\widetilde{N}}$ is the space of functions from $H^2(\mathbb{B}^n)$ orthogonal to $\mathcal{P}_{\widetilde{N}}$.

Taking into account the arguments which were given above we obtain that the method $\widehat{m}_{N',\lfloor M' \rfloor}(y_1, y_2)$ is the best for problem (5) and the method $\widehat{m}_{\widehat{N},\widehat{M}}(y_1, y_2)$ is the best for problem (1).

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