

# OPTIMAL RECOVERY OF SOLUTIONS OF THE GENERALIZED HEAT EQUATION IN THE UNIT BALL FROM INACCURATE DATA

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*We devote this paper to Henryk Woźniakowski who had intensively studied optimal recovery problems*

ABSTRACT. We consider the problem of optimal recovery of solutions of the generalized heat equation in the unit ball. Information is given at two time instances, but inaccurate. The solution is to be constructed at some intermediate time. We provide the optimal error and present an algorithm which achieves this error level.

The application of optimal recovery theory to problems of partial differential equations was started by J. F. Traub and H. Woźniakowski in [1]. In particular, this monograph considered optimal recovery of solutions of the heat equation from finitely many Fourier coefficients of the initial function. Several recovery problems for partial differential equation from noisy information were recently studied in [2]–[7]. The results considered in these papers were based on a general method for optimal recovery of linear operators developed in [8] and [9] (see also [10]). This method extended previous research from [11]. Various problems of optimal recovery from noisy information may be found in [12] (see also [13] where the complexity of differential and integral equations is discussed).

Here we consider the optimal recovery problem for solutions of the generalized heat equation in the unit  $d$ -ball at the time  $\tau$  from inaccurate solutions at the times  $t_1$  and  $t_2$ .

Set

$$\mathbb{B}^d = \left\{ x = (x_1, \dots, x_d) : |x|^2 = \sum_{j=1}^d x_j^2 < 1 \right\},$$
$$\mathbb{S}^{d-1} = \{ x \in \mathbb{R}^d : |x| = 1 \}.$$

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Consider the problem of finding the solution of the generalized heat equation in  $L_2(\mathbb{B}^d)$ :

$$(1) \quad \begin{aligned} u_t + (-\Delta)^{\alpha/2} u &= 0, \quad \alpha > 0, \\ u|_{t=0} &= f(x), \\ u|_{x \in \mathbb{S}^{d-1}} &= 0. \end{aligned}$$

Let  $0 \leq t_1 < t_2$ . Suppose we know approximate solutions  $y_1$  and  $y_2$  of (1) at times  $t_1$  and  $t_2$ , given with errors  $\delta_1$  and  $\delta_2$  in the  $L_2(\mathbb{B}^d)$  norm. We want to recover in the best way the solution of (1) at the time  $\tau$ ,  $t_1 < \tau < t_2$ . We assume that  $y_1, y_2 \in L_2(\mathbb{B}^d)$  satisfy

$$\|u(\cdot, t_j) - y_j(\cdot)\|_{L_2(\mathbb{B}^d)} \leq \delta_j, \quad j = 1, 2.$$

Any map  $\xi: L_2(\mathbb{B}^d) \times L_2(\mathbb{B}^d) \rightarrow L_2(\mathbb{B}^d)$  is admitted as a recovery method. The quantity

$$\begin{aligned} e_\tau(\alpha, L_2(\mathbb{B}^d), \delta_1, \delta_2, \xi) \\ = \sup_{\substack{f, y_1, y_2 \in L_2(\mathbb{B}^d) \\ \|u(\cdot, t_j) - y_j(\cdot)\|_{L_2(\mathbb{B}^d)} \leq \delta_j, \quad j=1,2}} \|u(\cdot, \tau) - \xi(y_1, y_2)(\cdot)\|_{L_2(\mathbb{B}^d)}, \end{aligned}$$

where  $u$  is the solution of (1), is called the error of the method  $\xi$ . The quantity

$$E_\tau(\alpha, L_2(\mathbb{B}^d), \delta_1, \delta_2) = \inf_{\xi: L_2(\mathbb{B}^d) \times L_2(\mathbb{B}^d) \rightarrow L_2(\mathbb{B}^d)} e_\tau(\alpha, L_2(\mathbb{B}^d), \delta_1, \delta_2, \xi)$$

is called the error of optimal recovery and a method delivering the lower bound is called an optimal recovery method.

Note that the initial functions  $f$  belong to the whole space  $L_2(\mathbb{B}^d)$ . In other words, the a priori information about initial functions is not a compact set. Therefore we use the information with infinite cardinality ([1] dealt with algorithms using information having finite cardinality). For example, it can be shown that knowing (even precisely) any finite number of Fourier coefficients of  $u(\cdot, t_j)$ ,  $j = 1, 2$ , does not lead to the finite error of optimal recovery.

The analysis of the problem is different for  $d = 1$  and  $d > 1$ , because of different type of orthogonal eigensystems.

We begin with the case  $d > 1$ . Let  $H_k$  denote the set of spherical harmonics of order  $k$ . It is known (see [14]) that  $\dim H_0 = a_0 = 1$ ,

$$\dim H_k = a_k = (d + 2k - 2) \frac{(d + k - 3)!}{(d - 2)!k!}, \quad k = 1, 2, \dots,$$

and

$$L_2(\mathbb{S}^{d-1}) = \sum_{k=0}^{\infty} H_k.$$

Let  $\{Y_j^{(k)}\}_{j=1}^{a_k}$  denote an orthonormal basis in  $H_k$ . Let  $J_p$  be the Bessel function of the first kind of order  $p$  and  $\mu_s^{(p)}$ ,  $s = 1, 2, \dots$ , be the zeros of  $J_p$ .

The functions

$$Z_{skj}(x) = \frac{J_p(\mu_s^{(p)} r)}{r^{d/2-1}} Y_j^{(k)}(x'),$$

where  $r = |x|$ ,  $x' = x/r$ , and  $p = k + (d - 2)/2$ , form an orthogonal basis in  $L_2(\mathbb{B}^d)$ . Moreover,

$$\Delta Z_{skj} = -(\mu_s^{(p)})^2 Z_{skj}.$$

We will use the orthonormal basis in  $L_2(\mathbb{B}^d)$

$$Y_{skj} = \frac{Z_{skj}}{\|Z_{skj}\|_{L_2(\mathbb{B}^d)}}.$$

We recall that the operator  $(-\Delta)^{\alpha/2}$  is defined as follows

$$(-\Delta)^{\alpha/2} f = \sum_{s=1}^{\infty} \sum_{k=0}^{\infty} (\mu_s^{(p)})^{\alpha} \sum_{j=1}^{a_k} c_{skj} Y_{skj},$$

where

$$(2) \quad f = \sum_{s=1}^{\infty} \sum_{k=0}^{\infty} \sum_{j=1}^{a_k} c_{skj} Y_{skj}.$$

The solution of (1) can be easily found by the Fourier method of separation of variables. It has the form

$$u(x, t) = \sum_{s=1}^{\infty} \sum_{k=0}^{\infty} e^{-(\mu_s^{(p)})^{\alpha} t} \sum_{j=1}^{a_k} c_{skj} Y_{skj}(x),$$

where  $c_{skj}$  are the Fourier coefficients of the initial function.

Set

$$a_{sk} = e^{-2(\mu_s^{(p)})^{\alpha}}$$

(we recall that  $p = k + (d - 2)/2$  and  $\alpha$  is from (1)). It is known (see [15]) that for all  $s \in \mathbb{N}$

$$\mu_s^{(p)} < \mu_s^{(p+1)} < \mu_{s+1}^{(p)}$$

and  $\mu_s^{(p)} \rightarrow \infty$  as  $s \rightarrow \infty$ . So the set of zeros of the Bessel functions  $\mu_s^{(p)}$ ,  $s = 1, 2, \dots$ ,  $p = k + (d - 2)/2$ ,  $k = 0, 1, \dots$ , can be arranged in ascending order

$$\mu_{s_1}^{(p_1)} < \mu_{s_2}^{(p_2)} < \dots < \mu_{s_n}^{(p_n)} < \dots$$

Consequently,

$$a_{s_1 k_1} > a_{s_2 k_2} > \dots > a_{s_n k_n} > \dots$$

For the case  $d = 1$  the functions

$$Y_s(x) = \sin \frac{\pi s}{2}(x + 1), \quad s = 1, 2, \dots,$$

form an orthonormal basis in  $L_2(\mathbb{B}^1) = L_2([-1, 1])$  and

$$\Delta Y_s = - \left( \frac{\pi s}{2} \right)^2 Y_s.$$

We define the operator  $(-\Delta)^{\alpha/2}$  as follows

$$(-\Delta)^{\alpha/2} f = \sum_{s=1}^{\infty} \left( \frac{\pi s}{2} \right)^{\alpha} c_s Y_s,$$

where  $c_s$  are the Fourier coefficients of  $f$ . It is easily verified that for  $d = 1$  the solution of (1) is given by

$$u(x, t) = \sum_{s=1}^{\infty} e^{-\left(\frac{\pi s}{2}\right)^{\alpha} t} c_s Y_s(x),$$

where  $c_s$  are the Fourier coefficients of the initial function.

For an arbitrary decreasing sequence  $\beta_1 > \beta_2 > \dots > 0$  we introduce the following notation

$$\begin{aligned} \Delta_m &= [\beta_{m+1}^{t_2-t_1}, \beta_m^{t_2-t_1}], \quad \Delta_0 = [\beta_1^{t_2-t_1}, +\infty), \\ \widehat{\lambda}_1 &= \begin{cases} \frac{\beta_{m+1}^{\tau-t_2} - \beta_m^{\tau-t_2}}{\beta_{m+1}^{t_1-t_2} - \beta_m^{t_1-t_2}}, & \frac{\delta_2^2}{\delta_1^2} \in \Delta_m, \quad m \geq 1, \\ \beta_1^{\tau-t_1}, & \frac{\delta_2^2}{\delta_1^2} \in \Delta_0, \end{cases} \\ \widehat{\lambda}_2 &= \begin{cases} \frac{\beta_m^{\tau-t_1} - \beta_{m+1}^{\tau-t_1}}{\beta_m^{t_2-t_1} - \beta_{m+1}^{t_2-t_1}}, & \frac{\delta_2^2}{\delta_1^2} \in \Delta_m, \quad m \geq 1, \\ 0, & \frac{\delta_2^2}{\delta_1^2} \in \Delta_0. \end{cases} \end{aligned}$$

**Theorem 1.** *Set*

$$\beta_m = \begin{cases} a_{s_m, k_m}, & d > 1, \\ e^{-2\left(\frac{\pi m}{2}\right)^{\alpha}}, & d = 1. \end{cases}$$

Then for all  $\delta_1, \delta_2 > 0$  the following equality

$$E_{\tau}(\alpha, L_2(\mathbb{B}^d), \delta_1, \delta_2) = \sqrt{\widehat{\lambda}_1 \delta_1^2 + \widehat{\lambda}_2 \delta_2^2}$$

holds. Moreover, the method

$$(3) \quad \widehat{\xi}(y_1, y_2)(x) = \begin{cases} \sum_{s=1}^{\infty} \sum_{k=0}^{\infty} a_{sk}^{\tau/2} \sum_{j=1}^{a_k} \frac{\widehat{\lambda}_1 a_{sk}^{t_1/2} y_{1skj} + \widehat{\lambda}_2 a_{sk}^{t_2/2} y_{2skj}}{\widehat{\lambda}_1 a_{sk}^{t_1} + \widehat{\lambda}_2 a_{sk}^{t_2}} Y_{skj}(x), & d > 1, \\ \sum_{s=1}^{\infty} e^{-\left(\frac{\pi s}{2}\right)^{\alpha} \tau} \frac{\widehat{\lambda}_1 e^{-\left(\frac{\pi s}{2}\right)^{\alpha} t_1} y_{1s} + \widehat{\lambda}_2 e^{-\left(\frac{\pi s}{2}\right)^{\alpha} t_2} y_{2s}}{\widehat{\lambda}_1 e^{-2\left(\frac{\pi s}{2}\right)^{\alpha} t_1} + \widehat{\lambda}_2 e^{-2\left(\frac{\pi s}{2}\right)^{\alpha} t_2}} Y_s(x), & d = 1, \end{cases}$$

where  $y_{1skj}, y_{2skj}$  and  $y_{1s}, y_{2s}$  are the Fourier coefficients of  $y_1(\cdot)$  and  $y_2(\cdot)$ , is optimal.

To prove Theorem 1 we use a general scheme of construction of optimal recovery methods for linear operators developed in [8] and [9] (see also [10]).

Consider the following extremal problem

$$(4) \quad \|u(\cdot, \tau)\|_{L_2(\mathbb{B}^d)}^2 \rightarrow \max, \quad \|u(\cdot, t_j)\|_{L_2(\mathbb{B}^d)}^2 \leq \delta_j^2, \quad j = 1, 2, \\ f \in L_2(\mathbb{B}^d),$$

where  $u$  is the solution of problem (1). Set

$$L(f, \lambda_1, \lambda_2) = -\|u(\cdot, \tau)\|_{L_2(\mathbb{B}^d)}^2 + \lambda_1 \|u(\cdot, t_1)\|_{L_2(\mathbb{B}^d)}^2 + \lambda_2 \|u(\cdot, t_2)\|_{L_2(\mathbb{B}^d)}^2.$$

From [9] (see also [10]) follows

**Theorem 2.** *Suppose that there exist  $\widehat{\lambda}_1 \geq 0, \widehat{\lambda}_2 \geq 0$  and an admissible function  $\widehat{f}$  in (4) such that*

$$(a) \quad \min_{f \in L_2(\mathbb{B}^d)} L(f, \widehat{\lambda}_1, \widehat{\lambda}_2) = L(\widehat{f}, \widehat{\lambda}_1, \widehat{\lambda}_2), \\ (b) \quad \widehat{\lambda}_1 (\|\widehat{u}(\cdot, t_1)\|_{L_2(\mathbb{B}^d)}^2 - \delta_1^2) + \widehat{\lambda}_2 (\|\widehat{u}(\cdot, t_2)\|_{L_2(\mathbb{B}^d)}^2 - \delta_2^2) = 0,$$

where  $\widehat{u}$  is the solution of (1) with the initial function  $\widehat{f}$ . If for all  $y_1, y_2 \in L_2(\mathbb{B}^d)$  there exists a solution  $f_0$  of the problem

$$\widehat{\lambda}_1 \|u(\cdot, t_1) - y_1(\cdot)\|_{L_2(\mathbb{B}^d)}^2 + \widehat{\lambda}_2 \|u(\cdot, t_2) - y_2(\cdot)\|_{L_2(\mathbb{B}^d)}^2 \rightarrow \min, \\ f \in L_2(\mathbb{B}^d),$$

where  $u$  is the solution of (1), then the method

$$\widehat{\xi}(y_1, y_2)(x) = u_0(x, \tau),$$

where  $u_0$  is the solution of (1) with the initial function  $f_0$ , is optimal and for the error of optimal recovery the following equality

$$E(\alpha, L_2(\mathbb{B}^d), \delta_1, \delta_2) = \sqrt{\widehat{\lambda}_1 \delta_1^2 + \widehat{\lambda}_2 \delta_2^2}$$

holds.

*Proof of Theorem 1.* Consider the case  $d > 1$ . We have

$$L(f, \widehat{\lambda}_1, \widehat{\lambda}_2) = \sum_{s=1}^{\infty} \sum_{k=0}^{\infty} (-a_{sk}^{\tau} + \widehat{\lambda}_1 a_{sk}^{t_1} + \widehat{\lambda}_2 a_{sk}^{t_2}) \sum_{j=1}^{a_k} c_{skj}^2,$$

where  $c_{skj}$  are the Fourier coefficients of  $f$ . Putting

$$b_{sk} = \sum_{j=1}^{a_k} c_{skj}^2,$$

we rewrite  $L(f, \widehat{\lambda}_1, \widehat{\lambda}_2)$  in the form

$$L(f, \widehat{\lambda}_1, \widehat{\lambda}_2) = \sum_{s=1}^{\infty} \sum_{k=0}^{\infty} a_{sk}^{\tau} (-1 + \widehat{\lambda}_1 a_{sk}^{t_1 - \tau} + \widehat{\lambda}_2 a_{sk}^{t_2 - \tau}) b_{sk}.$$

Assume that  $\delta_2^2 / \delta_1^2 \in \Delta_m$ ,  $m \geq 1$ . It is easily seen that in this case for  $\widehat{\lambda}_1$  and  $\widehat{\lambda}_2$ , equalities

$$(5) \quad \begin{aligned} \widehat{\lambda}_1 \beta_m^{t_1} + \widehat{\lambda}_2 \beta_m^{t_2} &= \beta_m^{\tau}, \\ \widehat{\lambda}_1 \beta_{m+1}^{t_1} + \widehat{\lambda}_2 \beta_{m+1}^{t_2} &= \beta_{m+1}^{\tau} \end{aligned}$$

hold. Consider the function  $g(z) = -1 + \widehat{\lambda}_1 e^{-2z(t_1 - \tau)} + \widehat{\lambda}_2 e^{-2z(t_2 - \tau)}$ . It is easy to verify that  $g$  is a convex function. It follows from (5) that  $g$  has two zeros  $z_m = (\mu_{s_m}^{(p_m)})^{\alpha}$  and  $z_{m+1} = (\mu_{s_{m+1}}^{(p_{m+1})})^{\alpha}$ . In view of the convexity of  $g$  for all  $z \leq z_m$  and all  $z \geq z_{m+1}$  the inequality  $g(z) \geq 0$  holds. Thus for all  $f \in L_2(\mathbb{B}^d)$  we have

$$L(f, \widehat{\lambda}_1, \widehat{\lambda}_2) \geq 0.$$

Define  $\widehat{b}_{s_m, k_m}$  and  $\widehat{b}_{s_{m+1}, k_{m+1}}$  from the conditions

$$\widehat{b}_{s_m, k_m} \beta_m^{t_j} + \widehat{b}_{s_{m+1}, k_{m+1}} \beta_{m+1}^{t_j} = \delta_j^2, \quad j = 1, 2.$$

It is easy to verify that

$$\begin{aligned} \widehat{b}_{s_m, k_m} &= \frac{\delta_1^2}{\beta_m^{t_1}} \cdot \frac{\delta_2^2 / \delta_1^2 - \beta_{m+1}^{t_2 - t_1}}{\beta_m^{t_2 - t_1} - \beta_{m+1}^{t_2 - t_1}}, \\ \widehat{b}_{s_{m+1}, k_{m+1}} &= \frac{\delta_1^2}{\beta_{m+1}^{t_1}} \cdot \frac{\beta_m^{t_2 - t_1} - \delta_2^2 / \delta_1^2}{\beta_m^{t_2 - t_1} - \beta_{m+1}^{t_2 - t_1}}. \end{aligned}$$

For  $j \neq m, m+1$  we set  $b_{s_j, k_j} = 0$ . Then the function

$$\widehat{f}(x) = \sum_{j=m}^{m+1} \sqrt{b_{s_j k_j}} Y_{s_j k_j 1}(x)$$

will be admissible and

$$L(\widehat{f}, \widehat{\lambda}_1, \widehat{\lambda}_2) = 0.$$

Thus conditions (a) and (b) of Theorem 2 hold.

Now we assume that  $\delta_2^2 / \delta_1^2 \in \Delta_0$ . It means that  $\delta_2^2 \geq \delta_1^2 \beta_1^{t_2 - t_1}$ . Putting

$$\widehat{f}(x) = \delta_1 \beta_1^{-t_1/2} Y_{s_1 k_1 1}(x),$$

for the solution  $\widehat{u}$  of (1) with the initial function  $\widehat{f}$  we have

$$\begin{aligned} \|\widehat{u}(\cdot, t_1)\|_{L_2(\mathbb{B}^d)}^2 &= \delta_1^2, \\ \|\widehat{u}(\cdot, t_2)\|_{L_2(\mathbb{B}^d)}^2 &= \delta_1^2 \beta_1^{t_2 - t_1} \leq \delta_2^2. \end{aligned}$$

Consequently, condition (b) of Theorem 2 holds. Condition (a) of the same theorem holds since for all functions  $f \in L_2(\mathbb{B}^d)$

$$L(f, \widehat{\lambda}_1, \widehat{\lambda}_2) \geq 0,$$

and moreover

$$L(\widehat{f}, \widehat{\lambda}_1, \widehat{\lambda}_2) = 0.$$

Now let us construct an optimal recovery method. According to Theorem 2 we have to solve the problem

$$\begin{aligned} & \widehat{\lambda}_1 \sum_{s=1}^{\infty} \sum_{k=0}^{\infty} \sum_{j=1}^{a_k} (a_{sk}^{t_1/2} c_{skj} - y_{1skj})^2 \\ & + \widehat{\lambda}_2 \sum_{s=1}^{\infty} \sum_{k=0}^{\infty} \sum_{j=1}^{a_k} (a_{sk}^{t_2/2} c_{skj} - y_{2skj})^2 \rightarrow \min, \quad f \in L_2(\mathbb{B}^d), \end{aligned}$$

where  $c_{skj}$  are the Fourier coefficients of  $f$  (see (2)). It can be easily verified that the solution of this problem has the form

$$\widehat{c}_{skj} = \frac{\widehat{\lambda}_1 a_{sk}^{t_1/2} y_{1skj} + \widehat{\lambda}_2 a_{sk}^{t_2/2} y_{2skj}}{\widehat{\lambda}_1 a_{sk}^{t_1} + \widehat{\lambda}_2 a_{sk}^{t_2}}.$$

The optimality of method (3) now follows from Theorem 2.

The case  $d = 1$  may be considered in a similar way.  $\square$

We give the table (see [15]) of the first ten ordered numbers  $\mu_s^{(p)}$  for even  $d$  (that is, for  $p \in \mathbb{Z}_+$ ) and for odd  $d$  (when  $p = k + 1/2$ ,  $k = 0, 1, \dots$ ).

$j$	$s_j$	$p_j$	$\mu_{s_j}^{(p_j)}$	$s_j$	$p_j$	$\mu_{s_j}^{(p_j)}$
1	1	0	2.4048	1	$\frac{1}{2}$	3.1416
2	1	1	3.8317	1	$\frac{3}{2}$	4.4934
3	1	2	5.1356	1	$\frac{5}{2}$	5.7635
4	2	0	5.5200	2	$\frac{1}{2}$	6.2832
5	1	3	6.3802	1	$\frac{7}{2}$	6.9879
6	2	1	7.0156	2	$\frac{3}{2}$	7.7253
7	1	4	7.5883	1	$\frac{9}{2}$	8.1826
8	2	2	8.4172	2	$\frac{5}{2}$	9.0950
9	3	0	8.6537	1	$\frac{11}{2}$	9.3558
10	1	5	8.7715	3	$\frac{1}{2}$	9.4248

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