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Optimal Recovery of Linear Operators from Inaccurate Information

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Lecture 1

1. Introduction

What does it mean to solve a problem in an optimal way? Assume that we have a problem p to be solved. Usually we have some information abut this problem. This information as a rule is incomplete and/or inaccurate. We denote it by $I(p)$. Suppose we have a method (algorithm) m to solve this problem. The method m uses the information $I(p)$. To compare the quality of different methods with each method m we have to associate a number indicating the error of the solution of the problem. We denote this number by $e(p, I, m)$.

Usually we want to have a method that can be applied to several problems of the same type. Assume that we have a set of problems $\mathcal{P}$. Then for the set $\mathcal{P}$ the error of the given method m may be defined as follows

$$e(\mathcal{P}, I, m) = \sup_{p\in\mathcal{P}} e(p, I, m).$$

If we want to find a good method for problems $\mathcal{P}$ we have to find a method for which the value $e(\mathcal{P}, I, m)$ as small as possible. Denote by $\mathcal{M}$ the set of admissible methods. Then we want to find a method $\widehat{m}$ such that

$$e(\mathcal{P}, I, \widehat{m}) = \inf_{m\in\mathcal{M}} e(\mathcal{P}, I, m) =: E(\mathcal{P}, I, \mathcal{M}).$$

We call the method $\widehat{m}$ an *optimal method* and the value $E(\mathcal{P}, I, \mathcal{M})$ is called an *optimal error*.

It may appears that $E(\mathcal{P}, I, \mathcal{M})$ is not sufficiently small. Then we may try to find another type of information about problems from $\mathcal{P}$ that can provide a better error of solutions. In other words, we can consider the following problem

$$\inf_{I\in\mathcal{I}} E(\mathcal{P}, I, \mathcal{M}),$$

where $\mathcal{I}$ is some set of information.

Let us consider some examples.

Example 1 (optimal interpolation)**.** Let W be some class of functions defined on a domain D. Denote by p_f the problem of finding $f(t)$, $t \in D$, for a function $f \in W$. Put

$$I(p_f) = I(f) = (f(t_1), \dots, f(t_n)), \quad t_j \in D,\ j = 1, \dots, n.$$

Let $\mathcal{M}$ be the set of all mappings $m\colon \mathbb{R}^n \to \mathbb{R}$. We put

$$e(p_f, I, m) = |f(t) - m(I(f))|.$$

Here $\mathcal{P} = \{p_f : f \in W\}$. Thus,

$$e(\mathcal{P}, I, m) = \sup_{f \in W} |f(t) - m(I(f))| =: e(t, W, I, m).$$

To find an optimal method we have to consider the following problem

$$E(t, W, I, \mathcal{M}) = \inf_{m \colon \mathbb{R}^n \to \mathbb{R}} e(t, W, I, m).$$

This problem is called the problem of optimal recovery of a function $f \in W$ at a fixed point t from the information about the values $f(t_1), \dots, f(t_n)$.

Example 2 (optimal integration)**.** Let p_f be the problem of finding the integral

$$Lf = \int_a^b f(t)\, dt$$

for a function $f \in W$. With the same $I(f)$, $\mathcal{P}$, and $\mathcal{M}$ we obtain the problem of optimal integration on the class W from the information about values of f at a fixed system of nodes

$$E(L, W, I, \mathcal{M}) = \inf_{m \colon \mathbb{R}^n \to \mathbb{R}} \sup_{f \in W} \left| \int_a^b f(t)\, dt - m(I(f)) \right|.$$

Note that if instead of $\mathcal{M}$ we consider the set $\mathcal{M}_0$ containing only linear functions m, that is,

$$m(I(f)) = \sum_{j=1}^{n} a_j f(t_j), \;\; a_j \in \mathbb{R}, \;\; j = 1, \dots, n,$$

then we obtain the well-known problem of finding optimal quadrature formula for the class W and a fixed system of nodes.

One may ask how to choose such points $a \le t_1 < \ldots < t_n \le b$ for which the optimal error will be minimal. In this case we obtain the following problem

$$E(L, W, \mathcal{I}, \mathcal{M}) = \inf_{I \in \mathcal{I}} E(L, W, I, \mathcal{M}),$$

where

$$\mathcal{I} = \{ \; I : a \le t_1 < \ldots < t_n \le b \; \}.$$

Example 3 (optimal numerical differentiation)**.** In notation of Example 1 this is the following problem

$$E'(t, W, I, \mathcal{M}) = \inf_{m \colon \mathbb{R}^n \to \mathbb{R}} \sup_{f \in W} |f'(t) - m(I(f))|.$$

Let us consider complete solutions of these problems for some simple classes.

2. Optimal interpolation for W^1_∞

Denote by W^1_∞ the class of real functions f defined on the interval $[-1, 1]$, absolutely continuous, and satisfying the condition

$$|f'(t)| \le 1 \quad \text{almost everywhere on } [-1, 1].$$

Following Example 1 we put

$$e(t, W^1_\infty, I_{\bar t}, m) = \sup_{f \in W^1_\infty} |f(t) - m(I_{\bar t}(f))|,$$
$$E(t, W^1_\infty, I_{\bar t}) = \inf_{m\colon \mathbb{R}^n \to \mathbb{R}} e(t, W^1_\infty, I_{\bar t}, m),$$

where

$$I_{\bar t}(f) = (f(t_1), \ldots, f(t_n)), \quad \bar t = (t_1, \ldots, t_n), \quad -1 \le t_1 < \ldots < t_n \le 1.$$

Denote by $\alpha(t)$ the nearest point to t from the set of nodes $\{t_1, \ldots, t_n\}$ (in the case when t is in the middle between t_j and t_{j+1} we set for definiteness $\alpha(t) = t_j$). Thus,

$$\alpha(t) = \begin{cases} t_1, & -1 \le t \le \dfrac{t_1 + t_2}{2}, \\ t_j, & \dfrac{t_{j-1} + t_j}{2} < t \le \dfrac{t_j + t_{j+1}}{2}, \ j = 2, \ldots, n-1, \\ t_n, & \dfrac{t_{n-1} + t_n}{2} < t \le 1. \end{cases}$$

Put

$$\widehat{f}(t) = |t - \alpha(t)|$$

(see Fig. 1).

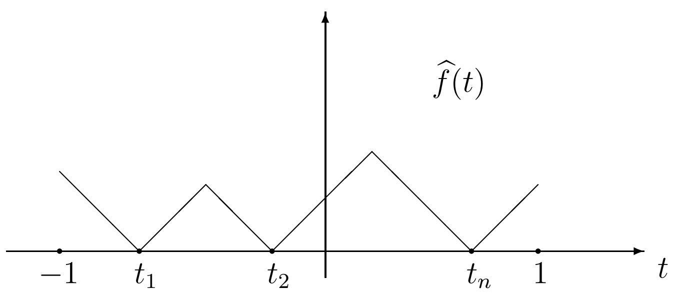

Figure 1

It is obvious that $\widehat{f} \in W^1_\infty$ and $-\widehat{f} \in W^1_\infty$. Moreover, $I_{\bar{t}}(\widehat{f}) = I_{\bar{t}}(-\widehat{f}) = 0$. For any method m we have

$$\begin{aligned}2\widehat{f}(t) &= |\widehat{f}(t) - m(0) - (-\widehat{f}(t) - m(0))| \\ &\le |\widehat{f}(t) - m(0)| + |-\widehat{f}(t) - m(0)| \le 2e(t, W^1_\infty, I_{\bar{t}}, m).\end{aligned}$$

Consequently, for all m

$$e(t, W^1_\infty, I_{\bar{t}}, m) \ge \widehat{f}(t).$$

Hence

$$E(t, W^1_\infty, I_{\bar{t}}) \ge \widehat{f}(t). \tag{1}$$

We obtain the lower bound. Now let us obtain the upper bound.

Define the method $\widehat{m}$ by the equality

$$\widehat{m}(I_{\bar{t}}(f)) = f(\alpha(t)).$$

Then

$$f(t) - f(\alpha(t)) = \int_{\alpha(t)}^{t} f'(\tau)\, d\tau.$$

Since $|f'(\tau)| \le 1$ we have

$$|f(t) - f(\alpha(t))| \le |t - \alpha(t)| = \widehat{f}(t).$$

Thus, for all $f \in W^1_\infty$

$$|f(t) - \widehat{m}(I_{\bar{t}}(f))| \le \widehat{f}(t).$$

We have

$$E(t, W^1_\infty, I_{\bar{t}}) \le e(t, W^1_\infty, I_{\bar{t}}, \widehat{m}) \le \widehat{f}(t).$$

Taking into account the lower bound (1), we obtain that

$$E(t, W^1_\infty, I_{\bar{t}}) = \widehat{f}(t)$$

and $\widehat{m}$ ia an optimal method. Consequently, if we have function values $f(t_1), \dots, f(t_n)$, then an optimal method of recovery of $f(t)$ on the class W^1_∞ is the following

$$f(t) \approx f(\alpha(t))$$

(see Fig. 2).

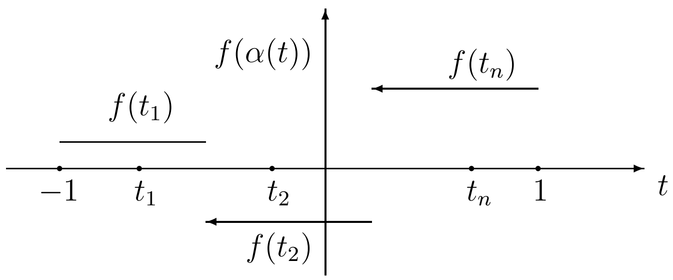

FIGURE 2

3. Optimal integration for W^1_∞

For the same class W^1_∞ and the same information $I_{\bar t}$ consider the problem of optimal recovery of the integral

$$Lf = \int_{-1}^{1} f(t)\,dt.$$

As in the previous example any functions $m\colon \mathbb{R}^n \to \mathbb{R}$ are admitted as recovery methods. The error of the method is defined as follows

$$e(L, W^1_\infty, I_{\bar t}, m) = \sup_{f \in W^1_\infty} \left| \int_{-1}^{1} f(t)\,dt - m(I_{\bar t}(f)) \right|.$$

We are interested in the optimal recovery error

$$E(L, W^1_\infty, I_{\bar t}) = \inf_{m\colon \mathbb{R}^n \to \mathbb{R}} e(L, W^1_\infty, I_{\bar t}, m)$$

and in an optimal method of recovery, that is, in the method for which the lower bound is attained. Using the same notation for the function $\widehat{f}(t) = |t - \alpha(t)|$ we obtain that for every method m

$$2\int_{-1}^{1} \widehat{f}(t)\,dt \le \left| \int_{-1}^{1} \widehat{f}(t)\,dt - m(0) \right| + \left| \int_{-1}^{1} (-\widehat{f}(t))\,dt - m(0) \right| \le 2e(L, W^1_\infty, I_{\bar t}, m).$$

Thus, for every method m

$$E(L, W^1_\infty, I_{\bar t}) \ge \int_{-1}^{1} \widehat{f}(t)\,dt. \tag{2}$$

To obtain the upper bound consider the method

$$\widehat{m}_0(I_{\bar t}(f)) = \int_{-1}^{1} f(\alpha(t))\,dt = \int_{-1}^{1} \widehat{m}(I_{\bar t})\,dt.$$

We can rewrite this method in the form

$$\widehat{m}_0(I_{\bar{t}}(f)) = \int_{-1}^{\frac{t_1+t_2}{2}} f(t_1)\,dt + \int_{\frac{t_1+t_2}{2}}^{\frac{t_2+t_3}{2}} f(t_2)\,dt + \ldots + \int_{\frac{t_{n-1}+t_n}{2}}^{1} f(t_n)\,dt$$

$$= \left(\frac{t_1+t_2}{2}+1\right) f(t_1) + \frac{t_3-t_1}{2} f(t_2) + \ldots + \left(1 - \frac{t_{n-1}+t_n}{2}\right) f(t_n).$$

We show that $\widehat{m}_0$ is an optimal method. We have

$$e(L, W_\infty^1, I_{\bar{t}}, \widehat{m}_0) = \sup_{f\in W_\infty^1} \left| \int_{-1}^{1} f(t)\,dt - \int_{-1}^{1} f(\alpha(t))\,dt \right|$$

$$\leq \sup_{f\in W_\infty^1} \int_{-1}^{1} |f(t) - f(\alpha(t))|\,dt \leq \int_{-1}^{1} |t-\alpha(t)|\,dt = \int_{-1}^{1} \widehat{f}(t)\,dt.$$

Thus,

$$E(L, W_\infty^1, I_{\bar{t}}) \leq \int_{-1}^{1} \widehat{f}(t)\,dt.$$

Taking into account the lower bound (2) we obtain that

$$E(L, W_\infty^1, I_{\bar{t}}) = \int_{-1}^{1} \widehat{f}(t)\,dt$$

and consequently the method $\widehat{m}_0$ is optimal.

Lecture 2

Let us try to find a system of nodes $-1 \le t_1^0 < \ldots < t_n^0 \le 1$ for which the error of optimal recovery will be minimal. In other words, we consider the extremal problem

$$\min_{-1 \le t_1 < \ldots < t_n \le 1} \int_{-1}^{1} \widehat{f}(t)\, dt.$$

We have to find $t_1 < \ldots < t_n$ to make the shaded area minimal (see Fig. 3).

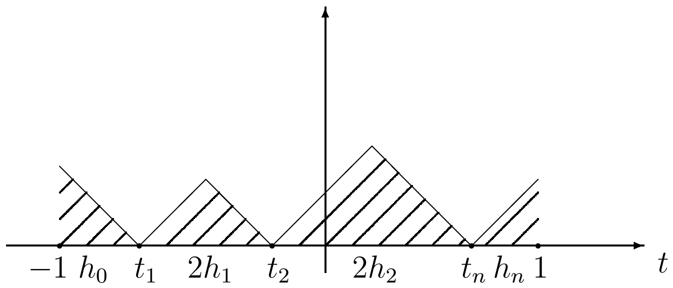

FIGURE 3

Put

$$h_0 = t_1 + 1, \quad 2h_j = t_{j+1} - t_j,\ j = 1, \ldots, n-1, \quad h_n = 1 - t_n.$$

Note that

$$h_0 + 2h_1 + \ldots + 2h_{n-1} + h_n = 2.$$

Then

$$\int_{-1}^{1} \widehat{f}(t)\, dt = \frac{h_0^2}{2} + h_1^2 + \ldots + h_{n-1}^2 + \frac{h_n^2}{2}.$$

We use the Cauchy-Shwartz inequality

$$\left|\sum_{j=1}^{r} a_j b_j\right| \le \sqrt{\sum_{j=1}^{r} a_j^2}\sqrt{\sum_{j=1}^{r} b_j^2}.$$

For $a_1 = \ldots = a_r = 1$ it gives

$$\sum_{j=1}^{r} b_j^2 \ge \frac{1}{r}\left(\sum_{j=1}^{r} b_j\right)^2.$$

Thus,

$$\int_{-1}^{1} \widehat{f}(t)\,dt = \frac{1}{2}(h_0^2 + (h_1^2 + h_1^2) + \ldots + (h_{n-1}^2 + h_{n-1}^2) + h_n^2)$$
$$\geq \frac{1}{2}\frac{(h_0 + 2h_1 + \ldots + 2h_{n-1} + h_n)^2}{2 + 2(n-1)} = \frac{1}{n}.$$

If we take $h_0 = h_1 = \ldots = h_n = 1/n$, then

$$\int_{-1}^{1} \widehat{f}(t)\,dt = \frac{1}{n}.$$

Consequently, the nodes

$$t_j^0 = -1 + \frac{2j-1}{n}, \quad j = 1, \ldots, n,$$

are optimal.

4. Optimal recovery of the derivative from inaccurate information

In the previous examples we use incomplete but exact information. Indeed we usually have some error in any input data. Let us consider the following problem with inaccurate information. We want to find approximate value of $f'(0)$ knowing approximate values of f at the points $-h$, h, $0 < h \leq 1$. We assume that

$$f \in W_\infty^2 = \{\, f : f' \in W_\infty^1 \,\}$$

and we know the values f_{-1}, f_1 such that

$$|f(-h) - f_{-1}| \leq \delta,$$
$$|f(h) - f_1| \leq \delta,$$

where $\delta > 0$ is the error of the input data. Any mapping $m\colon \mathbb{R}^2 \to \mathbb{R}$ is admitted as a recovery method. The error of the method m is defined as follows

$$e'(W_\infty^2, I_\delta^h, m) = \sup_{f \in W_\infty^2} \sup_{\substack{f_{-1}, f_1 \in \mathbb{R} \\ |f(jh) - f_j| \leq \delta,\ j=-1,1}} |f'(0) - m(f_{-1}, f_1)|.$$

We are interested in the error of optimal recovery

$$E'(W_\infty^2, I_\delta^h) = \inf_{m\colon \mathbb{R}^2 \to \mathbb{R}} e'(W_\infty^2, I_\delta^h, m)$$

and in an optimal method of recovery, that is, a method for which the lower bound is attained.

Put

$$\widehat{f}(t)=\begin{cases}-\dfrac{t^2}{2}+\left(\dfrac{h}{2}+\dfrac{\delta}{h}\right)t, & 0\le t\le 1,\\ \dfrac{t^2}{2}+\left(\dfrac{h}{2}+\dfrac{\delta}{h}\right)t, & -1\le t\le 0.\end{cases}$$

1. The lower bound. It is easily verified that $\widehat{f},-\widehat{f}\in W^2_\infty$ and $\widehat{f}(-h)=-\delta$, $\widehat{f}(h)=\delta$. For any method m we have

$$2\widehat{f}'(0)\le|\widehat{f}'(0)-m(0,0)|+|-\widehat{f}'(0)-m(0,0)|\le 2e'(W^2_\infty,I^h_\delta,m).$$

Consequently,

$$E'(W^2_\infty,I^h_\delta)\ge\widehat{f}'(0)=\frac{h}{2}+\frac{\delta}{h}.$$

2. The upper bound. Consider the method

$$\widehat{m}(f_{-1},f_1)=\frac{f_1-f_{-1}}{2h}.$$

Taking into account that $f_j=f(jh)+\delta_j$, $j=-1,1$, we have

(3)
$$\begin{aligned}e'(W^2_\infty,I^h_\delta,\widehat{m})&=\sup_{f\in W^2_\infty}\ \sup_{|\delta_j|\le\delta,\ j=-1,1}\left|f'(0)-\frac{f(h)+\delta_1-f(-h)-\delta_{-1}}{2h}\right|\\&\le\sup_{f\in W^2_\infty}\left|f'(0)-\frac{f(h)-f(-h)}{2h}\right|+\frac{\delta}{h}.\end{aligned}$$

To estimate the last supremum we need the following

Lemma 1. *If $f\in W^2_\infty$, then for all $\tau\in[-1,1]$ there exists $M\in[-1,1]$ such that*

$$f(\tau)=f(0)+f'(0)\tau+M\frac{\tau^2}{2}. \tag{4}$$

Proof. We have

$$\int_0^\tau f''(t)(\tau-t)\,dt=\int_0^\tau(\tau-t)\,df'(t)=-\tau f'(0)+f(\tau)-f(0).$$

Since $f\in W^2_\infty$ we obtain

$$\left|\int_0^\tau f''(t)(\tau-t)\,dt\right|\le\left|\int_0^\tau|\tau-t|\,dt\right|=\frac{\tau^2}{2}.$$

□

Using (4) for $\tau = h$ and $\tau = -h$, we have

$$f(h) = f(0) + f'(0)h + M_1\frac{h^2}{2},$$

$$f(-h) = f(0) - f'(0)h + M_{-1}\frac{h^2}{2}.$$

Hence

$$f'(0) = \frac{f(h) - f(-h)}{2h} - (M_1 - M_{-1})\frac{h}{4}.$$

Consequently, for $f \in W_\infty^2$

$$\left| f'(0) - \frac{f(h) - f(-h)}{2h} \right| \le \frac{h}{2}.$$

Now it follows from (3) that

$$e'(W_\infty^2, I_\delta^h, \widehat{m}) \le \frac{h}{2} + \frac{\delta}{h} = \widehat{f}'(0).$$

Taking into account the lower bound, we obtain that

$$E'(W_\infty^2, I_\delta^h) = \frac{h}{2} + \frac{\delta}{h}$$

and method $\widehat{m}$ is optimal.

Lecture 3

Consider the problem of optimization of input information for the value

$$E'(W^2_\infty, I^h_\delta) = \frac{h}{2} + \frac{\delta}{h}.$$

It is easy to see (see Fig.4) that this function (as a function of x) has the unique minimum on the interval $(0, 1]$ at the point

$$\widehat{h} = \begin{cases} \sqrt{2\delta}, & 0 < \delta < \dfrac{1}{2}, \\ 1, & \delta \geq \dfrac{1}{2}. \end{cases}$$

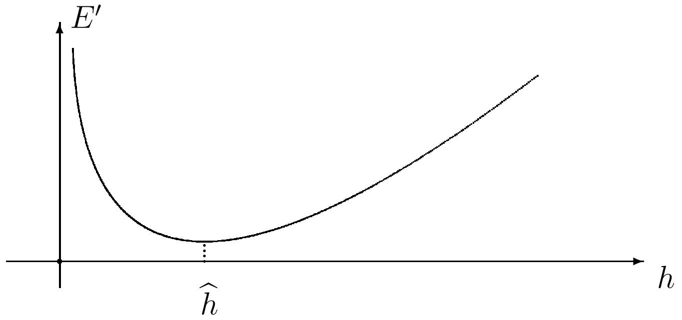

FIGURE 4

Thus,

$$\min_{0<h\leq 1} E'(W^2_\infty, I^h_\delta) = \begin{cases} \sqrt{2\delta}, & 0 < \delta < \dfrac{1}{2}, \\ \dfrac{1}{2} + \delta, & \delta \geq \dfrac{1}{2}. \end{cases}$$

5. Recovery of a function at a point from inaccurate information

Denote by $L_2(\mathbb{R})$ the space of functions f defined on $\mathbb{R}$ for which

$$\|f\|_{L_2(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(t)|^2 \, dt \right)^{1/2} < \infty.$$

Let $\mathcal{W}^1_2(\mathbb{R})$ be the space of locally absolutely continuous functions $f \in L_2(\mathbb{R})$ for which $\|f'\|_{L_2(\mathbb{R})} < \infty$. We denote by W^1_2 the class of functions $f \in \mathcal{W}^1_2(\mathbb{R})$ for which $\|f'\|_{L_2(\mathbb{R})} \leq 1$. For the class W^1_2 we consider the problem of optimal recovery of the value $f(0)$ from the information about the function f given with the error $\delta > 0$ in the $L_2(\mathbb{R})$-norm. We assume that for each function $f \in W_2^1$ we know a function $y \in L_2(\mathbb{R})$ such that

$$\|f - y\|_{L_2(\mathbb{R})} \le \delta.$$

Knowing y we have to obtain a best possible approximation to the value $f(0)$.

Similar to the previous examples we are interested in the optimal recovery error

$$E_0(W_2^1, I_\delta^{\mathbb{R}}) = \inf_{m\colon L_2(\mathbb{R}) \to \mathbb{R}} e_0(W_2^1, I_\delta^{\mathbb{R}}, m),$$

where

$$e_0(W_2^1, I_\delta^{\mathbb{R}}, m) = \sup_{f \in W_2^1} \sup_{\substack{y \in L_2(\mathbb{R}) \\ \|f-y\|_{L_2(\mathbb{R})} \le \delta}} |f(0) - m(y)|,$$

and in optimal method of recovery (that is, in a method for which the infimum is attained).

1. The lower bound. Let $m\colon L_2(\mathbb{R}) \to \mathbb{R}$ be an arbitrary method, $f \in W_2^1$, and $\|f\|_{L_2(\mathbb{R})} \le \delta$. Then

$$2|f(0)| \le |f(0) - m(0)| + |-f(0) - m(0)| \le 2e_0(W_2^1, I_\delta^{\mathbb{R}}, m).$$

Thus,

$$e_0(W_2^1, I_\delta^{\mathbb{R}}, m) \ge |f(0)|.$$

Taking the infimum over all methods m and then the supremum over all functions $f \in W_2^1$ such that $\|f\|_{L_2(\mathbb{R})} \le \delta$, we obtain

$$E_0(W_2^1, I_\delta^{\mathbb{R}}) \ge \sup_{\substack{f \in W_2^1 \\ \|f\|_{L_2(\mathbb{R})} \le \delta}} |f(0)|.$$

It is easy to check that the function

$$\widehat{f}(t) = \sqrt{\delta} e^{-|t|/\delta}$$

belongs to the class W_2^1 and $\|\widehat{f}\|_{L_2(\mathbb{R})} = \delta$. Consequently,

$$E_0(W_2^1, I_\delta^{\mathbb{R}}) \ge |\widehat{f}(0)| = \sqrt{\delta}. \tag{5}$$

2. The upper bound. First we prove that for all $f \in W_2^1$

$$|f(t)| \le \sqrt{|t|}\|f'\|_{L_2(\mathbb{R})} + |f(0)|.$$

Using the Cauchy-Shwartz inequality, we have

$$|f(t) - f(0)| = \left| \int_0^t f'(t)\,dt \right| \le \sqrt{|t|}\|f'\|_{L_2(\mathbb{R})}.$$

Thus,

$$|f(t)| \le |f(t) - f(0)| + |f(0)| \le \sqrt{|t|}\|f'\|_{L_2(\mathbb{R})} + |f(0)|.$$

Consequently,

$$\lim_{t\to\pm\infty} f(t)e^{-|t|/\delta} = 0.$$

To find optimal method of recovery we prove that for all $f \in W_2^1$ the following identity

$$f(0) = \frac{1}{2\delta}\int_{\mathbb{R}} e^{-|t|/\delta} f(t)\,dt - \frac{1}{2}\int_{\mathbb{R}} e^{-|t|/\delta} f'(t)\operatorname{sign} t\,dt \tag{6}$$

holds. We have

$$\int_0^\infty e^{-t/\delta} f'(t)\,dt = \int_0^\infty e^{-t/\delta}\,df(t) = e^{-t/\delta} f(t)\Big|_0^\infty + \frac{1}{\delta}\int_0^\infty f(t)e^{-t/\delta}\,dt$$
$$= -f(0) + \frac{1}{\delta}\int_0^\infty f(t)e^{-t/\delta}\,dt.$$

Thus,

$$f(0) = \frac{1}{\delta}\int_0^\infty f(t)e^{-t/\delta}\,dt - \int_0^\infty e^{-t/\delta} f'(t)\,dt.$$

In a similar way we obtain, that

$$f(0) = \frac{1}{\delta}\int_{-\infty}^0 f(t)e^{t/\delta}\,dt + \int_{-\infty}^0 e^{t/\delta} f'(t)\,dt.$$

Adding these two equalities we obtain that (6) holds.

Now let us estimate the error of the method

$$\widehat{m}(y) = \frac{1}{2\delta}\int_{\mathbb{R}} e^{-|t|/\delta} y(t)\,dt.$$

Assume that $f \in W_2^1$, $y \in L_2(\mathbb{R})$, and $\|f-y\|_{L_2(\mathbb{R})} \le \delta$. Then

$$|f(0) - \widehat{m}(y)| = \left| f(0) - \frac{1}{2\delta}\int_{\mathbb{R}} e^{-|t|/\delta}(y(t) - f(t) + f(t))\,dt \right|$$
$$\le \left| f(0) - \frac{1}{2\delta}\int_{\mathbb{R}} e^{-|t|/\delta} f(t)\,dt \right| + \frac{1}{2\delta}\left| \int_{\mathbb{R}} e^{-|t|/\delta}(y(t) - f(t))\,dt \right|.$$

Using (6) and the Cauchy-Shwartz inequality, we have

$$|f(0) - \widehat{m}(y)| \le \frac{1}{2}\int_{\mathbb{R}} e^{-|t|/\delta}|f'(t)|\,dt + \frac{1}{2}\sqrt{\int_{\mathbb{R}} e^{-2|t|/\delta}\,dt}$$
$$\le \sqrt{\int_{\mathbb{R}} e^{-2|t|/\delta}\,dt} = \frac{\|\widehat{f}\|_{L_2(\mathbb{R})}}{\sqrt{\delta}} = \sqrt{\delta}.$$

Hence

$$E_0(W_2^1, I_\delta^{\mathbb{R}}) \le e_0(W_2^1, I_\delta^{\mathbb{R}}, \widehat{m}) \le \sqrt{\delta}.$$

Taking into account the lower bound (5), we obtain

$$E_0(W_2^1, I_\delta^{\mathbb{R}}) = \sup_{\substack{f \in W_2^1 \\ \|f\|_{L_2(\mathbb{R})} \le \delta}} |f(0)| = \sqrt{\delta}.$$

Moreover, the method $\widehat{m}$ is an optimal method of recovery.

Lecture 4

Let x be an arbitrary function from $\mathcal{W}_2^1(\mathbb{R})$ such, that $x \neq const$. Put
$$f = \frac{x}{\|x'\|_{L_2(\mathbb{R})}},$$
then $f \in W_\infty^2$. Set
$$\delta = \frac{\|x\|_{L_2(\mathbb{R})}}{\|x'\|_{L_2(\mathbb{R})}},$$
then $\|f\|_{L_2(\mathbb{R})} = \delta$. Since
$$|f(0)| \leq \sup_{\substack{f \in W_2^1 \\ \|f\|_{L_2(\mathbb{R})} \leq \delta}} |f(0)| = \sqrt{\delta},$$
we have
$$\frac{|x(0)|}{\|x'\|_{L_2(\mathbb{R})}} \leq \frac{\|x\|_{L_2(\mathbb{R})}^{1/2}}{\|x'\|_{L_2(\mathbb{R})}^{1/2}}.$$
Thus,
$$|x(0)| \leq \|x\|_{L_2(\mathbb{R})}^{1/2} \|x'\|_{L_2(\mathbb{R})}^{1/2}. \tag{7}$$
This is one of the so-called inequalities of Landau–Kolmogorov type. These inequalities play a significant role in optimal recovery problems. On the other hand, inequality (7) may be considered as an uncertainty principal. It stays that both the norm of the function and the norm of the derivative could not be sufficiently small at the same time.

6. General setting

Let X be a linear space, Z be a normed linear space, and $T\colon X \to Z$ be a linear operator. We consider the problem of optimal recovery of the operator T on a set $W \subset X$ from the information about many-valued operator $F\colon W \to Y$ (for each $x \in W$, $F(x)$ is a set from Y). We assume that for every $x \in W$ we know an element $y \in F(x)$. Knowing y we have to approximate the value Tx. Every mapping $m\colon Y \to Z$ is admitted as a recovery method (or an algorithm) (see Fig. 5).

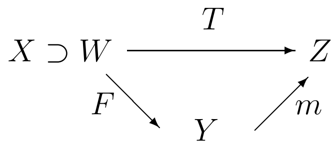

Figure 5

For a given method m we define the *error of the method* as follows

$$e(T,W,F,m)=\sup_{x\in W}\sup_{y\in F(x)}\|Tx-m(y)\|_Z.$$

The quantity

$$E(T,W,F)=\inf_{m\colon Y\to Z}e(T,W,F,m)$$

is called the *error of optimal recovery.*

Lemma 2 (the lower bound)**.** *Assume that the set*

$$F^{-1}(0)=\{\,x\in W:F(x)=0\,\}$$

is not empty and centrally-symmetric (that is, for any $x\in F^{-1}(0)$, $-x\in F^{-1}(0)$). Then

$$E(T,W,F)\geq\sup_{x\in F^{-1}(0)}\|Tx\|_Z.$$

Proof. Let $x\in F^{-1}(0)$ and m be an arbitrary method of recovery. Then since $-x\in F^{-1}(0)$ we have

$$\begin{aligned}2\|Tx\|_Z&=\|Tx-m(0)-(-Tx-m(0))\|_Z\\&\leq\|Tx-m(0)\|_Z+\|-Tx-m(0)\|_Z\leq 2e(T,W,F,m).\end{aligned}$$

Taking the supremum over all $x\in F^{-1}(0)$ we obtain that for all $m\colon Y\to Z$

$$e(T,W,F,m)\geq\sup_{x\in F^{-1}(0)}\|Tx\|_Z.$$

Consequently,

$$E(T,W,F)=\inf_{m\colon Y\to Z}e(T,W,F,m)\geq\sup_{x\in F^{-1}(0)}\|Tx\|_Z.$$

□

Now we consider the problem of optimal recovery of linear operators for linear spaces with semi-inner products. Recall that Y is a linear space with a semi-inner product $(\cdot,\cdot)_Y$, if there exists a mapping which associates with every pair $x,y\in X$ a real (or, in general, complex) number $(x,y)_Y$ such, that

1. $(x,x)_Y\geq 0.$
2. $(x,y)_Y=\overline{(y,x)}_Y.$
3. $(\alpha x+\beta y,z)_Y=\alpha(x,z)_Y+\beta(y,z)_Y,\quad \alpha,\beta\in\mathbb{C}.$

Let X be a linear space, $Y_1,\dots,Y_n$ be linear spaces with semi-inner products $(\cdot,\cdot)_{Y_j}$, $j=1,\dots,n$, and the corresponding semi-norms $\|\cdot\|_{Y_j}$ ($\|x\|_{Y_j}=\sqrt{(x,x)_{Y_j}}$), $I_j\colon X\to Y_j$, $j=1,\dots,n$, be linear operators, and Z be a normed linear space. We consider the problem of optimal recovery of the operator $T\colon X \to Z$ on the set

$$W_k = \{\, x \in X : \|I_j x\|_{Y_j} \le \delta_j,\ 1 \le j \le k,\ 0 \le k < n \,\}$$

(for $k = 0$ we take $W = X$) from the information about values of operators $I_{k+1}, \dots, I_n$ given with errors. We assume that for any $x \in W$ we know the vector $y = (y_{k+1}, \dots, y_n)$ such that

$$\|I_j x - y_j\|_{Y_j} \le \delta_j, \quad j = k+1, \dots, n$$

Knowing the vector y we want to recover Tx.

Using the notation of the general setting, in this problem we have

$$F(x) = \{\, y = (y_{k+1}, \dots, y_n) \in Y_{k+1} \times \ldots \times Y_n : \\ \|I_j x - y_j\|_{Y_j} \le \delta_j,\ j = k+1, \dots, n \,\}.$$

Any operator $m\colon Y_{k+1} \times \ldots \times Y_n \to Z$ is admitted as a recovery method. According to the general setting the value

$$e(T, W_k, I, \delta, m) = \sup_{x \in W_k} \sup_{\substack{y = (y_{k+1}, \dots, y_n) \in Y_{k+1} \times \ldots \times Y_n \\ \|I_j x - y_j\|_{Y_j} \le \delta_j,\ j = k+1, \dots, n}} \|Tx - m(y)\|_Z$$

is called the error of recovery of the method m (here $I = (I_1, \dots, I_n)$, $\delta = (\delta_1, \dots \delta_n)$). The quantity

$$E(T, W_k, I, \delta) = \inf_{m\colon Y_{k+1} \times \ldots \times Y_n \to Z} e(T, W_k, I, \delta, m) \tag{8}$$

is called the error of optimal recovery. A method delivering the lower bound is called optimal.

The formulated problem of optimal recovery is closely connected with the following extremal problem (we shall call it the *duality* extremal problem)

$$\|Tx\|_Z^2 \to \max, \quad \|I_j x\|_{Y_j}^2 \le \delta_j^2,\ j = 1, \dots, n,\ x \in X. \tag{9}$$

Lecture 5

Now we formulate the main result. It what follows we will apply it to many problems of optimal recovery.

Theorem 1. *Assume that there exist* $\widehat{\lambda}_j \geq 0$, $j = 1, \ldots, n$, *such that the value of the extremal problem*

$$\|Tx\|_Z^2 \to \max, \quad \sum_{j=1}^n \widehat{\lambda}_j \|I_j x\|_{Y_j}^2 \leq \sum_{j=1}^n \widehat{\lambda}_j \delta_j^2, \quad x \in X \tag{10}$$

is the same as in (9)*. Moreover, assume that for all* $y = (y_1, \ldots, y_n) \in Y_1 \times \ldots \times Y_n$ *there exists* $x_y = x(y_1, \ldots, y_n)$ *which is a solution of the extremal problem*

$$\sum_{j=1}^n \widehat{\lambda}_j \|I_j x - y_j\|_{Y_j}^2 \to \min, \quad x \in X. \tag{11}$$

Then for all k, $0 \leq k < n$,

$$E(T, W_k, I, \delta) = \sup_{\substack{x \in X \\ \|I_j x\|_{Y_j} \leq \delta_j,\ j=1,\ldots,n}} \|Tx\|_Z$$

and the method

$$\widehat{m}(y_{k+1}, \ldots, y_n) = Tx(0, \ldots, 0, y_{k+1}, \ldots, y_n) \tag{12}$$

is optimal.

To prove this theorem we need a preliminary result concerning a best approximation property in a linear space with a semi-inner product. Let Y be a linear space with a semi-inner product $(\cdot, \cdot)_Y$ and L be a subspace of Y. Let $y \in Y$. Consider the problem of best approximation of y by elements from L

$$\|x - y\|_Y \to \min, \quad x \in L. \tag{13}$$

Proposition 1. *If* $\widehat{x}$ *is a solution of* (13)*, then for all* $x \in L$

$$(\widehat{x} - y, x)_Y = 0.$$

Proof. Suppose that there exists $x_0 \in L$ such that

$$(\widehat{x} - y, x_0)_Y = \alpha \neq 0.$$

Put $z=\widehat{x}-\lambda x_0$, where $\lambda=\alpha/\|x_0\|_Y^2$. Note that $z\in L$. We have

$$\begin{aligned}\|z-y\|_Y^2 &= (\widehat{x}-\lambda x_0-y,\widehat{x}-\lambda x_0-y)_Y\\ &= \|\widehat{x}-y\|_Y^2-2\,\mathrm{Re}(\widehat{x}-y,\lambda x_0)_Y+|\lambda|^2\|x_0\|_Y^2\\ &= \|\widehat{x}-y\|_Y^2-2\,\mathrm{Re}(\overline{\lambda}\alpha)+\frac{|\alpha|^2}{\|x_0\|_Y^2}=\|\widehat{x}-y\|_Y^2-\frac{|\alpha|^2}{\|x_0\|_Y^2}<\|\widehat{x}-y\|_Y^2.\end{aligned}$$

This contradiction proves the assertion of the theorem. □

Proof of Theorem 1. The lower bound. Since

$$F^{-1}(0)=\{\,x\in W:\|I_jx\|_{Y_j}\le\delta_j,\ j=k+1,\dots,n\,\}.$$

from Lemma 2 we have

$$E(T,W_k,I,\delta)\ge\sup_{\substack{x\in W\\ \|I_jx\|_{Y_j}\le\delta_j,\ j=k+1,\dots,n}}\|Tx\|_Z=\sup_{\substack{x\in X\\ \|I_jx\|_{Y_j}\le\delta_j,\ j=1,\dots,n}}\|Tx\|_Z. \tag{14}$$

The upper bound. Consider the linear space $E=Y_1\times\ldots\times Y_n$ with the semi-inner product

$$(y^1,y^2)_E=\sum_{j=1}^n\widehat{\lambda}_j(y_j^1,y_j^2)_{Y_j},$$

where $y^1=(y_1^1,\dots,y_n^1)$, $y^2=(y_1^2,\dots,y_n^2)$. Now the extremal problem (11) can be rewritten in the form

$$\|\widetilde{I}x-y\|_E^2\to\max,\quad x\in X,$$

where $\widetilde{I}x=(I_1x,\dots,I_nx)$ and $y=(y_1,\dots,y_n)$. It follows from Proposition 1 that for all $x\in X$

$$(\widetilde{I}x_y-y,\widetilde{I}x)_E=0.$$

Consequently,

$$\|\widetilde{I}x-y\|_E^2=\|\widetilde{I}x-\widetilde{I}x_y\|_E^2+\|\widetilde{I}x_y-y\|_E^2.$$

Indeed, we have

$$\begin{aligned}\|\widetilde{I}x-y\|_E^2 &= \|\widetilde{I}x-\widetilde{I}x_y+\widetilde{I}x_y-y\|_E^2\\ &= \|\widetilde{I}x-\widetilde{I}x_y\|_E^2-2\,\mathrm{Re}(\widetilde{I}x-\widetilde{I}x_y,\widetilde{I}x_y-y)_E+\|\widetilde{I}x_y-y\|_E^2\\ &= \|\widetilde{I}x-\widetilde{I}x_y\|_E^2+\|\widetilde{I}x_y-y\|_E^2.\end{aligned}$$

Thus, for all $x\in X$

$$\|\widetilde{I}x-\widetilde{I}x_y\|_E^2\le\|\widetilde{I}x-y\|_E^2=\sum_{j=1}^n\widehat{\lambda}_j\|I_jx-y_j\|_{Y_j}^2. \tag{15}$$

Let $x \in X$ and $y = (0, \dots, 0, y_{k+1}, \dots, y_n)$ such that $\|I_j x - y_j\|_{Y_j} \le \delta_j$, $j = k+1, \dots, n$. Put $z = x - x_y$. Then it follows from (15) that

$$\sum_{j=1}^{n} \widehat{\lambda}_j \|I_j z\|_{Y_j}^2 = \|\widetilde{I} z\|_E^2 \le \sum_{j=1}^{n} \widehat{\lambda}_j \delta_j^2.$$

Now for the method (12) we have the following estimate

$$\begin{aligned} \|Tx - \widehat{m}(0, \dots, 0, y_1, \dots, y_n)\|_Z^2 = \|Tz\|_Z^2 \\ \le \sup_{\substack{z \in X \\ \sum_{j=1}^{n} \widehat{\lambda}_j \|I_j z\|_{Y_j}^2 \le \sum_{j=1}^{n} \widehat{\lambda}_j \delta_j^2}} \|Tz\|_Z^2 \\ = \sup_{\substack{x \in X \\ \|I_j x\|_{Y_j} \le \delta_j,\ j=1,\dots,n}} \|Tx\|_Z^2. \end{aligned}$$

Consequently,

$$E(T, W_k, I, \delta) \le e(T, W_k, I, \delta, \widehat{m}) \le \sup_{\substack{x \in X \\ \|I_j x\|_{Y_j} \le \delta_j,\ j=1,\dots,n}} \|Tx\|_Z.$$

Taking into account the lower bound (14), we obtain that

$$E(T, W_k, I, \delta) = \sup_{\substack{x \in X \\ \|I_j x\|_{Y_j} \le \delta_j,\ j=1,\dots,n}} \|Tx\|_Z$$

and $\widehat{m}$ is an optimal method. $\square$

Lecture 6

Now we obtain a sufficient conditions for coinciding the values of problems (9) and (10). Put

$$\mathcal{L}(x,\lambda) = -\|Tx\|_Z^2 + \sum_{j=1}^n \lambda_j \|I_j x\|_{Y_j}^2$$

(here $\lambda = (\lambda_1, \dots, \lambda_n)$. $\mathcal{L}$ is the so-called the Lagrange function for the extremal problem (9). We call $\widehat{x} \in X$ an extremal element if it is admissible in (9) (that is, $\|I_j\widehat{x}\|_{Y_j}^2 \le \delta_j^2$) and

$$\|T\widehat{x}\|_Z^2 = \sup_{\substack{x\in X\\ \|I_jx\|_{Y_j}^2\le\delta_j^2,\ j=1,\dots,n}} \|Tx\|_Z^2.$$

Theorem 2 (sufficient condition)**.** *Assume that there exist* $\widehat{\lambda}_j \ge 0$*,* $j = 1, \dots, n$*, and* $\widehat{x} \in X$ *admissible in* (9) *such that*

$$(a) \quad \min_{x\in X} \mathcal{L}(x,\widehat{\lambda}) = \mathcal{L}(\widehat{x},\widehat{\lambda}), \quad \widehat{\lambda} = (\widehat{\lambda}_1, \dots, \widehat{\lambda}_n),$$

$$(b) \quad \sum_{j=1}^n \widehat{\lambda}_j(\|I_j\widehat{x}\|_{Y_j}^2 - \delta_j^2) = 0.$$

Then $\widehat{x}$ *is an extremal element and*

$$\sup_{\substack{x\in X\\ \|I_jx\|_{Y_j}^2\le\delta_j^2,\ j=1,\dots,n}} \|Tx\|_Z^2 = \sup_{\substack{x\in X\\ \sum_{j=1}^n \widehat{\lambda}_j\|I_jx\|_{Y_j}^2\le\sum_{j=1}^n\widehat{\lambda}_j\delta_j^2}} \|Tx\|_Z^2 = \sum_{j=1}^n \widehat{\lambda}_j\delta_j^2.$$

Proof. Set

$$S = \sum_{j=1}^n \widehat{\lambda}_j\delta_j^2.$$

Let $x \in X$ be an admissible element in (9). Then

$$\begin{aligned} -\|Tx\|_Z^2 &\ge -\|Tx\|_Z^2 + \sum_{j=1}^n \widehat{\lambda}_j(\|I_jx\|_{Y_j}^2 - \delta_j^2) = \mathcal{L}(x,\widehat{\lambda}) - S \\ &\ge \mathcal{L}(\widehat{x},\widehat{\lambda}) - S = -\|T\widehat{x}\|_Z^2 + \sum_{j=1}^n \widehat{\lambda}_j(\|I_j\widehat{x}\|_{Y_j}^2 - \delta_j^2) = -\|T\widehat{x}\|_Z^2. \end{aligned}$$

The same arguments show that $\widehat{x}$ is an extremal element in the problem (10).

Now we prove that $\mathcal{L}(\widehat{x}, \widehat{\lambda}) = 0$. Suppose that $\mathcal{L}(\widehat{x}, \widehat{\lambda}) = a > 0$. Consider $x_0 = \alpha\widehat{x}$, $\alpha < 1$. We have

$$\mathcal{L}(x_0, \widehat{\lambda}) = \alpha^2 \mathcal{L}(\widehat{x}, \widehat{\lambda}) < \mathcal{L}(\widehat{x}, \widehat{\lambda}).$$

If $a < 0$, we put $\alpha > 1$. Then again

$$\mathcal{L}(x_0, \widehat{\lambda}) = \alpha^2 \mathcal{L}(\widehat{x}, \widehat{\lambda}) < \mathcal{L}(\widehat{x}, \widehat{\lambda}).$$

Consequently,

$$\sup_{\substack{x \in X \\ \|I_j x\|^2_{Y_j} \le \delta_j^2,\ j=1,\dots,n}} \|Tx\|^2_Z = \|T\widehat{x}\|^2_Z = -\mathcal{L}(\widehat{x}, \widehat{\lambda}) + S = S.$$

□

7. Optimal recovery of derivatives

Assume that we have the Fourier series for some 2π-periodic function x:

$$x(t) = \sum_{j=-\infty}^{+\infty} x_j e^{ijt}.$$

Suppose that we know only a finite number of Fourier coefficients which are given with an error. That is, we know $\tilde{x}_j$, $|j| \le N$, such that

$$|x_j - \tilde{x}_j| \le \delta, \quad |j| \le N. \tag{16}$$

Using the information $\{\tilde{x}_j\}_{|j|\le N}$ we want to recover the k-th derivative of x.

One of the simplest methods of recovery is the following

$$x^{(k)}(t) \approx \sum_{|j|\le N} (ij)^k \tilde{x}_j e^{ijt}.$$

But it is not very good because for large j the error of terms $(ij)^k \tilde{x}_j$ may be large. Since

$$|(ij)^k x_j - (ij)^k \tilde{x}_j| \le j^k \delta$$

it may be of order $j^k \delta$.

In practice this effect are known very well. Those who deal with such problems simply cut the terms with high frequencies or smooth them by some filter.

The problem which we would like to pose is: what is a best method of recovery? Or, in other words, what is a best possible filter? Now we give the exact setting of the problem.

Denote by $\mathbb{T}$ the unit circle realized as the interval $[-\pi,\pi]$ with identified endpoints. We denote by $L_2(\mathbb{T})$ the set of square integrable functions x on $\mathbb{T}$ with norm

$$\|x\|_{L_2(\mathbb{T})} = \left(\frac{1}{2\pi}\int_{\mathbb{T}} |x(t)|^2\,dt\right)^{1/2}.$$

The space $\mathcal{W}_2^r(\mathbb{T})$ is the set of 2π-periodic functions x for which the $(r-1)$-st derivative is absolutely continuous and $\|x^{(r)}\|_{L_2(\mathbb{T})} < \infty$. The class $W_2^r(\mathbb{T})$ is the set of 2π-periodic functions from $\mathcal{W}_2^r(\mathbb{T})$ for which $\|x^{r)}\|_{L_2(\mathbb{T})} \le 1$.

We assume that for every function $x \in W_2^r(\mathbb{T})$ we know the numbers $\tilde{x}_j$, $|j| < n$, such that (16) is fulfilled. The problem is to find the value

$$E_\infty^N(D^k, W_2^r(\mathbb{T}), \delta) = \inf_{m\colon \mathbb{C}^{2N+1}\to L_2(\mathbb{T})} \sup_{\substack{x\in W_2^r(\mathbb{T}),\ \tilde{x}=\{\tilde{x}_j\}_{|j|\le N}\\ |x_j-\tilde{x}_j|\le\delta,\ |j|\le N}} \|x^{(k)} - m(\tilde{x})\|_{L_2(\mathbb{T})}$$

and a corresponding optimal method of recovery (that is, the method delivering the lower bound).

Using notation of the general setting, here $X = \mathcal{W}_2^r(\mathbb{T})$, $Z = L_2(\mathbb{T})$, $Tx = D^k x = x^{(k)}$, $Y_1 = L_2(\mathbb{T})$, $Y_2 = \ldots = Y_{2N+2} = \mathbb{C}$, $I_1 x = x^{(r)}$, $I_j x = x_{-N+j-2}$, $j = 2,\ldots,2N+2$, $\delta_1 = 1$, $\delta_2 = \ldots = \delta_{2N+2} = \delta$,

$$W = \{\, x \in X : \|I_1 x\|_{Y_1} \le \delta_1 \,\}.$$

Consider the dual problem

$$(17)\quad \|x^{(k)}\|^2_{L_2(\mathbb{T})} \to \max,\quad \|x^{(r)}\|^2_{L_2(\mathbb{T})} \le 1,\quad |x_j|^2 \le \delta^2,\ |J| \le N,\quad x \in \mathcal{W}_2^r(\mathbb{T}).$$

The Lagrange function for this problem has the form

$$\mathcal{L}(x,\bar{\lambda}) = -\|x^{(k)}\|^2_{L_2(\mathbb{T})} + \lambda\|x^{(r)}\|^2_{L_2(\mathbb{T})} + \sum_{|j|\le N} \lambda_j |x_j|^2,$$

where $\bar{\lambda} = (\lambda, \lambda_{-N}, \ldots, \lambda_N)$. Since for all $0 \le s \le r$

$$x^{(s)}(t) = \sum_{j=-\infty}^{+\infty} (ij)^s x_j e^{ijt},$$

we have

$$\|x^{(s)}\|_{L_2(\mathbb{T})} = \sum_{j=-\infty}^{+\infty} j^{2s}|x_j|^2.$$

Thus,

$$\mathcal{L}(x,\bar\lambda)=\sum_{|j|\le N}(-j^{2k}+\lambda j^{2r}+\lambda_j)|x_j|^2+\sum_{|j|>N}(-j^{2k}+\lambda j^{2r})|x_j|^2.$$

It follows from Theorem 2 that it is sufficiently to find an admissible element $\widehat{x}\in\mathcal{W}_2^r(\mathbb{T})$ and $\bar{\widehat{\lambda}}=(\widehat{\lambda},\widehat{\lambda}_{-N},\dots,\widehat{\lambda}_N)$ such that conditions (a) and (b) of this theorem will be fulfilled and then to find a solution of extremal problem (11).

Lecture 7

Set

$$p_0 = \max\{\, p \in \mathbb{Z}_+ : \delta^2 \sum_{|j|<p} j^{2r} < 1, \quad 0 \le p \le N \,\}. \tag{18}$$

Put

$$\widehat{\lambda} = \frac{1}{(p_0+1)^{2(r-k)}}, \quad \widehat{\lambda}_j = \begin{cases} j^{2k} - \widehat{\lambda} j^{2r}, & |j| \le p_0, \\ 0, & p_0 + 1 \le |j| \le N, \end{cases}$$

$$\widehat{x}_j = \begin{cases} \delta, & |j| \le p_0, \\ \dfrac{1}{\sqrt{2}(p_0+1)^r} \sqrt{1 - \delta^2 \sum\limits_{|s|\le p_0} s^{2r}}, & |j| = p_0 + 1, \\ 0, & |j| > p_0 + 1. \end{cases}$$

Let us prove that

$$\widehat{x}(t) = \sum_{|s|\le p_0+1} \widehat{x}_j e^{ist}$$

is admissible function in extremal problem (17). We have

$$\|\widehat{x}^{(r)}\|^2_{L_2(\mathbb{T})} = \delta^2 \sum_{|s|\le p_0} s^{2r} + 1 - \delta^2 \sum_{|s|\le p_0} s^{2r} = 1.$$

It remains to prove that if $p_0 < N$, then $|\widehat{x}_j| \le \delta$. Suppose that

$$\frac{1}{2(p_0+1)^{(2r)}} \left(1 - \delta^2 \sum_{|s|\le p_0} s^{2r}\right) > \delta^2.$$

It means that

$$\delta^2 \sum_{|s|<p_0+1} s^{2r} < 1.$$

This contradicts the definition of p_0.

Since

$$\mathcal{L}(x, \overline{\overline{\lambda}}) = \sum_{|j|>p_0+1} (-j^{2k} + \widehat{\lambda} j^{2r})|x_j|^2 \ge 0$$

and $\mathcal{L}(\widehat{x}, \overline{\overline{\lambda}}) = 0$, condition (a) of Theorem 2 is fulfilled. We obtained that $\|\widehat{f}^{2r)}\|_{L_2(\mathbb{T})} = 1$. Together with equalities $|\widehat{x}_j| = \delta$, $|j| \le p_0 + 1$, it gives that condition (b) of the same theorem is fulfilled, too.

Consider the extremal problem (11). It has the following form

$$\widehat{\lambda}\|x^{(r)}\|^2_{L_2(\mathbb{T})} + \sum_{|j|\le p_0} \widehat{\lambda}_j |x_j - \tilde{x}_j|^2 \to \min, \quad x \in \mathcal{W}^r_2(\mathbb{T}).$$

We rewrite it in the form

$$\sum_{|j|\le p_0}(\widehat{\lambda}_j|x_j-\tilde{x}_j|^2+\widehat{\lambda}j^{2r}|x_j|^2)+\widehat{\lambda}\sum_{|j|>p_0}j^{2r}|x_j|^2\to\min,\quad x\in\mathcal{W}_2^r(\mathbb{T}).$$

Obviously, the solution of this problem is

$$x_j^0=\begin{cases}\dfrac{\widehat{\lambda}_j}{\widehat{\lambda}_j+\widehat{\lambda}j^{2r}}\tilde{x}_j, & |j|\le p_0,\\ 0, & |j|>p_0.\end{cases}$$

It follows from Thorem 1 that the method

$$\widehat{m}(\tilde{x})=(x^0)^{(k)}(t)=\sum_{|j|\le p_0}(ij)^k x_j^0 e^{ijt}$$

is optimal. Thus, we proved the following

Theorem 3. *Let $k,r\in\mathbb{Z}_+$, $0\le k<r$, $N\in\mathbb{N}$, $\delta>0$, and p_0 be defined by* (18). *Then*

$$E_\infty^N(D^k,W_2^r(\mathbb{T}),\delta)=\sqrt{\frac{1}{(p_0+1)^{2(r-k)}}+\delta^2\sum_{|j|\le p_0}\alpha_j j^{2k}},$$

where

$$\alpha_j=1-\left(\frac{j}{p_0+1}\right)^{2(r-k)}.$$

Moreover, the method

$$\widehat{m}(\tilde{x})=\sum_{|j|\le p_0}(ij)^k\alpha_j\tilde{x}_j e^{itj}$$

is optimal.

Note that α_j are monotonically decreasing as j various from 0 to p_0. It means that the optimal method $\widehat{m}$ smooths approximate values of Fourier coefficients $\tilde{x}_j$ for large j.

Lecture 8

Consider some arguments which explain how to find $\widehat{\lambda}$, $\widehat{\lambda}_j$, $|j| \le N$, and $\widehat{x}$. First, note that

$$-j^{2k} + \widehat{\lambda} j^{2r} + \widehat{\lambda}_j \ge 0,\ |j| \le N, \quad -j^{2k} + \widehat{\lambda} j^{2r} \ge 0,\ |j| > N.$$

Indeed, assume that for some s such that $|s| \le N$

$$-s^{2k} + \widehat{\lambda} s^{2r} + \widehat{\lambda}_s < 0.$$

Put

$$\widehat{x}_j = \begin{cases} c, & j = s, \\ 0, & j \ne s. \end{cases}$$

Then

$$\mathcal{L}(\widehat{x}, \overline{\widehat{\lambda}}) = (-s^{2k} + \widehat{\lambda} s^{2r} + \widehat{\lambda}_s)|c|^2 < 0.$$

In this case $\mathcal{L}(\widehat{x}, \overline{\widehat{\lambda}}) \to -\infty$ as $c \to \infty$. Consequently,

$$\min_{x \in \mathcal{W}_2^r(\mathbb{T})} \mathcal{L}(x, \overline{\widehat{\lambda}}) = -\infty.$$

The case $|s| > N$ may be considered in a similar way.

Since $\mathcal{L}(\widehat{x}, \overline{\widehat{\lambda}}) = 0$ we have

$$(-j^{2k} + \widehat{\lambda} j^{2r} + \widehat{\lambda}_j)|\widehat{x}_j| = 0,\ |j| \le N, \quad (-j^{2k} + \widehat{\lambda} j^{2r})|\widehat{x}_j| = 0,\ |j| > N.$$

It follows from condition (b) that if $\widehat{\lambda}_j \ne 0$, then $|\widehat{x}_j| = \delta$ and consequently, $-j^{2k} + \widehat{\lambda} j^{2r} + \widehat{\lambda}_j = 0$. Suppose we take $\widehat{x}_j = \delta$, $|j| \le p$, then since $\widehat{x} \in W_2^r(\mathbb{T})$ we have

$$\delta^2 \sum_{|j| \le p} j^{2r} \le 1.$$

Note also that $\widehat{\lambda} \ne 0$ otherwise $-j^{2k} + \widehat{\lambda} j^{2r} < 0$, $|j| > p$. Thus, we need to choose $\widehat{x}$ such that $\|\widehat{x}^{(2r)}\|_{L_2(\mathbb{T})} = 1$. All these arguments lead to the right choice of $\widehat{\lambda}$, $\widehat{\lambda}_j$, $|j| \le N$, and $\widehat{x}$.

Let $\delta > 0$ be a fixed number. If $p_0 < N$, then the further increase of the number of Fourier coefficients known with the same error δ does not decrease the error of optimal recovery. Thus for the fixed δ the system of $2N(\delta) + 1$ Fourier coefficients (or $2N(\delta)$ coefficients for the case $k > 0$, since in this case the zero coefficient is not used in the optimal method $\widehat{m}$), where

$$N(\delta) = \max\left\{ N \in \mathbb{Z}_+ : \delta^2 \sum_{|j| \le N} j^{2r} < 1 \right\},$$

allows to recover $x^{(k)}$ with the best possible accuracy.

Set $\delta_0 = \infty$,

$$\delta_s = \left(\sum_{|j| \le s} j^{2r} \right)^{-1/2}, \quad s = 1, 2, \dots .$$

Then for $\delta \in [\delta_{s+1}, \delta_s)$, $s = 0, 1, \dots,$ $N(\delta) = s$.
Let $r = 2$ and $k = 1$. Then

$$E_\infty^N(D, W_2^2(\mathbb{T}), \delta) = \frac{1}{(p_0+1)} \sqrt{1 + \delta^2 \sum_{|j| \le p_0} (j^2(p_0+1)^2 - j^4)}.$$

Using equalities

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6},$$

$$\sum_{j=1}^{n} j^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}, \tag{19}$$

which may be easily proved by induction, we obtain

$$E_\infty^N(D, W_2^2(\mathbb{T}), \delta) = \frac{1}{p_0+1} \sqrt{1 + \delta^2 \frac{p_0(p_0+1)(p_0+2)(2p_0+1)(2p_0+3)}{15}}. \tag{20}$$

If $k = 0$, then

$$E_\infty^N(D^0, W_2^2(\mathbb{T}), \delta) = \frac{1}{(p_0+1)^2} \sqrt{1 + \delta^2 \sum_{|j| \le p_0} ((p_0+1)^4 - j^4)} = \frac{1}{(p_0+1)^2} \sqrt{1 + \delta^2 \frac{(p_0+1)(2p_0+1)(12p_0^3 + 42p_0^2 + 46p_0 + 15)}{15}}. \tag{21}$$

We give some values of function $N(\delta)$ and the corresponding optimal recovery errors.

δ^2	$N(\delta)$	$(E_\infty^{N(\delta)}(D, W_2^r(\mathbb{T}), \delta))^2$	$(E_\infty^{N(\delta)}(D^0, W_2^r(\mathbb{T}), \delta))^2$
$\left[\frac{1}{2}, +\infty\right)$	0	1	$1+\delta^2$
$\left[\frac{1}{34}, \frac{1}{2}\right)$	1	$\frac{1+6\delta^2}{4}$	$\frac{1+46\delta^2}{16}$
$\left[\frac{1}{196}, \frac{1}{34}\right)$	2	$\frac{1+56\delta^2}{9}$	$\frac{1+361\delta^2}{81}$
$\left[\frac{1}{708}, \frac{1}{196}\right)$	3	$\frac{1+252\delta^2}{16}$	$\frac{1+1596\delta^2}{256}$

It may be directly verified that for $n \geq 1$

$$6\left(n+\frac{1}{3}\right)^5 < n(n+1)(2n+1)(3n^2+3n-1) < 6\left(n+\frac{1}{2}\right)^5.$$

It follows from (19) that

$$\frac{2}{5}\left(N(\delta)+\frac{1}{3}\right)^5 < \sum_{|j|\leq N(\delta)} j^4 < \frac{2}{5}\left(N(\delta)+\frac{1}{2}\right)^5.$$

In view of the definition of $N(\delta)$ we have

$$\left(\sum_{|j|\leq N(\delta)+1} j^4\right)^{-1/2} \leq \delta < \left(\sum_{|j|\leq N(\delta)} j^4\right)^{-1/2}.$$

Thus,

$$\frac{2}{5}\left(N(\delta)+\frac{3}{2}\right)^5 < \delta^{-2} < \frac{2}{5}\left(N(\delta)+\frac{1}{3}\right)^5.$$

Using these inequalities we obtain

$$\left(\frac{5}{2\delta^2}\right)^{1/5} - \frac{3}{2} < N(\delta) < \left(\frac{5}{2\delta^2}\right)^{1/5} - \frac{1}{3}.$$

Now from (20) and (21) we have

$$E_\infty^{N(\delta)}(D, W_2^2(\mathbb{T}), \delta) = \sqrt{\frac{7}{6}}\left(\frac{2\delta^2}{5}\right)^{1/5} + O(\delta^{4/5}),$$

$$E_\infty^{N(\delta)}(D^0, W_2^2(\mathbb{T}), \delta) = \sqrt{5}\left(\frac{2\delta^2}{5}\right)^{1/5} + O(\delta^{4/5}).$$

Lecture 9

Now we consider the case when approximate values of Fourier coefficients $\tilde{x}_j$ satisfy the condition

$$\sum_{j=-\infty}^{+\infty} |x_j - \tilde{x}_j|^2 \le \delta^2.$$

We define the error of optimal recovery as follows

$$E_2(D^k, W_2^r(\mathbb{T}), \delta) = \inf_{m\colon l_2 \to L_2(\mathbb{T})} \sup_{\substack{x \in W_2^r(\mathbb{T}),\ \tilde{x} = \{\tilde{x}_j\}_{j\in\mathbb{Z}} \in l_2 \\ \sum_{j=-\infty}^{+\infty} |x_j - \tilde{x}_j|^2 \le \delta^2}} \|x^{(k)} - m(\tilde{x})\|_{L_2(\mathbb{T})},$$

where l_2 is the space of vectors $\{x_j\}_{j\in\mathbb{Z}}$ such that

$$\sum_{j=-\infty}^{+\infty} |x_j|^2 < \infty.$$

Now the duality problem has the form

$$\text{(22)} \quad \|x^{(k)}\|_{L_2(\mathbb{T})}^2 \to \max, \quad \|x^{(r)}\|_{L_2(\mathbb{T})}^2 \le 1, \quad \sum_{j=-\infty}^{+\infty} |x_j|^2 \le \delta^2, \quad x \in \mathcal{W}_2^r(\mathbb{T}).$$

Consider the Lagrange function for this extremal problem

$$\begin{aligned}\mathcal{L}(x, \lambda_1, \lambda_2) &= -\|x^{(k)}\|_{L_2(\mathbb{T})}^2 + \lambda_1 \|x^{(r)}\|_{L_2(\mathbb{T})}^2 + \lambda_2 \sum_{j=-\infty}^{+\infty} |x_j|^2 \\ &= \sum_{j=-\infty}^{+\infty} (-j^{2k} + \lambda_1 j^{2r} + \lambda_2)|x_j|^2 \\ &= \sum_{j=-\infty}^{+\infty} -j^{2k}(-1 + \lambda_1 j^{2(r-k)} + \lambda_2 j^{-2k})|x_j|^2.\end{aligned}$$

Consider the function

$$F(x) = -1 + \lambda_1 x^{2(r-k)} + \lambda_2 x^{-2k}, \quad x > 0.$$

It is easily verified that $f(x)$ is a convex function. Thus, if $f(s) = f(s+1) = 0$, $s \ge 1$, then for all $j \ge 1$, $f(j) \ge 0$.

For fixed $s \geq 1$ we find $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$ from the condition $f(s) = f(s+1) = 0$. We have

$$\begin{aligned} \widehat{\lambda}_1 s^{2(r-k)} + \widehat{\lambda}_2 s^{-2k} &= 1, \\ \widehat{\lambda}_1 (s+1)^{2(r-k)} + \widehat{\lambda}_2 (s+1)^{-2k} &= 1. \end{aligned}$$

Hence,

$$\begin{aligned} \widehat{\lambda}_1 &= \frac{(s+1)^{2k} - s^{2k}}{(s+1)^{2r} - s^{2r}}, \\ \widehat{\lambda}_2 &= \frac{(s+1)^{2r} s^{2k} - s^{2r}(s+1)^{2k}}{(s+1)^{2r} - s^{2r}}. \end{aligned}$$

It may be easily checked that $\widehat{\lambda}_1, \widehat{\lambda}_2 \geq 0$. Thus, we have

$$\mathcal{L}(x, \widehat{\lambda}_1, \widehat{\lambda}_2) \geq 0.$$

Put

$$\widehat{x}(t) = \widehat{x}_s e^{ist} + \widehat{x}_{s+1} e^{i(s+1)t}. \tag{23}$$

Then

$$\|\widehat{x}^{(r)}\|^2_{L_2(\mathbb{T})} = |\widehat{x}_s|^2 s^{2r} + |\widehat{x}_{s+1}|^2 (s+1)^{2r}.$$

To satisfy the conditions

$$\|\widehat{x}^{(r)}\|^2_{L_2(\mathbb{T})} = 1, \quad \sum_{j=-\infty}^{+\infty} |\widehat{x}_j|^2 = \delta^2 \tag{24}$$

we should have

$$\begin{aligned} |\widehat{x}_s|^2 s^{2r} + |\widehat{x}_{s+1}|^2 (s+1)^{2r} &= 1, \\ |\widehat{x}_s|^2 + |\widehat{x}_{s+1}|^2 &= \delta^2. \end{aligned}$$

It follows from these equations that

$$\begin{aligned} |\widehat{x}_s|^2 &= \frac{\delta^2 (s+1)^{2r} - 1}{(s+1)^{2r} - s^{2r}}, \\ |\widehat{x}_{s+1}|^2 &= \frac{1 - \delta^2 s^{2r}}{(s+1)^{2r} - s^{2r}}. \end{aligned}$$

Thus, for

$$\frac{1}{(s+1)^r} \leq \delta < \frac{1}{s^r}$$

$\widehat{x}$ is admissible function in (22) and $\mathcal{L}(\widehat{x}, \widehat{\lambda}_1, \widehat{\lambda}_2) = 0$.

If $\delta \geq 1$ we put wl_1=1 and $\widehat{\lambda}_2 = 0$. Then

$$\mathcal{L}(x, 1, 0) = \sum_{j=-\infty}^{+\infty} j^{2k}(-1 + j^{2(r-k)})|x_j|^2 \geq 0.$$

Let $\widehat{x} = e^{it}$. Then $\mathcal{L}(\widehat{x}, 1, 0) = 0$. Moreover,

$$\|\widehat{x}^{(r)}\|_{L_2(\mathbb{T})} = 1, \qquad \sum_{j=-\infty}^{+\infty} |\widehat{x}_j|^2 = 1 \leq \delta^2.$$

Consequently, $\widehat{x}$ is admissible function.

Now it follows from Theorems 2 and 1 that in order to find an optimal method of recovery we have to solve the following extremal problem

$$\widehat{\lambda}_1 \|x^{(r)}\|^2_{L_2(\mathbb{T})} + \widehat{\lambda}_2 \sum_{j=-\infty}^{+\infty} |x_j - \tilde{x}_j|^2 \to \min, \quad x \in \mathcal{W}_2^r(\mathbb{T}).$$

Rewriting this problem in the form

$$\sum_{j=-\infty}^{+\infty} (\widehat{\lambda}_1 j^{2r} |x_j|^2 + \widehat{\lambda}_2 |x_j - \tilde{x}_j|^2) \to \min, \quad x \in \mathcal{W}_2^r(\mathbb{T}),$$

we can easily find the solution of this problem

$$x_j^0 = \frac{\widehat{\lambda}_2}{\widehat{\lambda}_2 + j^{2r}\widehat{\lambda}_1} \tilde{x}_j.$$

It follows from Theorem 1 that the method

$$\widehat{m}(\tilde{x}) = \sum_{j=-\infty}^{+\infty} (ij)^k \frac{\widehat{\lambda}_2}{\widehat{\lambda}_2 + j^{2r}\widehat{\lambda}_1} \tilde{x}_j e^{ijt}$$

is optimal for the considered problem. Thus we prove the following result.

Theorem 4. *Let $k, n \in \mathbb{N}$, $0 < k < n$, and $\delta > 0$. Then for*

$$\frac{1}{(s+1)^r} \leq \delta < \frac{1}{s^r}, \quad s = 1, 2, \dots ,$$

$$E_2(D^k, W_2^r(\mathbb{T}), \delta) = \sqrt{\delta^2 s^{2k} + (1 - \delta^2 s^{2r}) \frac{(s+1)^{2k} - s^{2k}}{(s+1)^{2r} - s^{2r}}}.$$

Moreover, the method

$$\widehat{m}(\tilde{x}) = \sum_{j=-\infty}^{+\infty} (ij)^k \left(1 + j^{2r} \frac{(s+1)^{2k} - s^{2k}}{s^{2k}(s+1)^{2r} - (s+1)^{2k} s^{2r}}\right)^{-1} \tilde{x}_j e^{ijt}$$

is optimal. For $\delta \geq 1$, $E_2(D^k, W_2^r(\mathbb{T}), \delta) = 1$ *and the method* $\widehat{m}(\tilde{x}) = 0$ *is optimal.*

Lecture 10

Consider again the following question: how to find $\widehat{\lambda}_1$, $\widehat{\lambda}_2$, and $\widehat{x}$ for the Lagrange function of the dual problem. We give now a graphical illustration which helps to answer this question.

Recall that the Lagrange function may be written in the following form

$$\mathcal{L}(x, \lambda_1, \lambda_2) = \sum_{j=-\infty}^{+\infty} (-j^{2k} + \lambda_1 j^{2r} + \lambda_2)|x_j|^2.$$

Consider the set of points on the plane $\mathbb{R}^2$

$$\begin{cases} x_j = j^{2r}, \\ y_j = j^{2k}, \end{cases} \quad j = 0, 1, \dots \ . \tag{25}$$

If we plot the function

$$\begin{cases} x = t^{2r}, \\ y = t^{2k}, \end{cases} \quad t \in [0, +\infty), \tag{26}$$

then the points (25) belong to the plot of this function. The function defined by (26) can be written in the form

$$y = x^{k/r}, \quad 0 < \frac{k}{r} < 1.$$

It is a convex function. Consequently, the piecewise linear function passing through the points (25) is also convex.

Let $s^{2r} < \delta^{-2} \le (s+1)^{2r}$. Assume that the line $y = \widehat{\lambda}_1 x + \widehat{\lambda}_2$ passes through the points (s^{2r}, s^{2k}) and $((s+1)^{2r}, (s+1)^{2k})$. Then in view of convexity for all points (j^{2r}, j^{2k}), $j = 0, 1, \dots$,

$$j^{2k} \le y(j^{2r}) = \widehat{\lambda}_1 j^{2r} + \widehat{\lambda}_2.$$

It means that $-j^{2k} + \widehat{\lambda}_1 j^{2r} + \widehat{\lambda}_2 \ge 0$. Thus, for all $x \in \mathcal{W}_2^r(\mathbb{T})$, $\mathcal{L}(x, \widehat{\lambda}_1, \widehat{\lambda}_2) \ge 0$.

Taking $\widehat{x}_j = 0$, $j \ne s, s+1$, and choosing $\widehat{x}_s$ and $\widehat{x}_{s+1}$ from the condition (24), we obtain that $\widehat{x}$ defined by (23) is admissible function and $\mathcal{L}(\widehat{x}, \widehat{\lambda}_1, \widehat{\lambda}_2) = 0$. Hence,

$$\min_{x \in \mathcal{W}_2^r(\mathbb{T})} \mathcal{L}(x, \widehat{\lambda}_1, \widehat{\lambda}_2) = \mathcal{L}(\widehat{x}, \widehat{\lambda}_1, \widehat{\lambda}_2).$$

By the way,

$$\widehat{\lambda}_1 \delta^{-2} + \widehat{\lambda}_2 = \frac{1}{\delta^2}(\widehat{\lambda}_1 + \widehat{\lambda}_2 \delta^2) = \frac{1}{\delta^2}(E_2(D^k, W_2^r(\mathbb{T}), \delta))^2.$$

Thus,

$$(E_2(D^k, W_2^r(\mathbb{T}), \delta))^2 = \frac{y(\delta^{-2})}{\delta^{-2}}.$$

The last value is the tangent of the angle between the line connected the origin with the point $(\delta^{-2}, y(\delta^{-2}))$ and the axis Ox.

We see that in this problem in optimal recovery method (for the case when $\delta < 1$) we use all information about approximate values of Fourier coefficients. It appears that we can construct another optimal recovery method that will use only a finite number of inaccurate Fourier coefficients.

Consider the case when we know approximate values of the Fourier coefficients $\tilde{x}_j$, $|j| \le N$, such that

$$\sum_{|j|\le N} |x_j - \tilde{x}_j|^2 \le \delta^2.$$

In this case the duality problem has the form

$$\|x^{(k)}\|^2_{L_2(\mathbb{T})} \to \max, \quad \|x^{(r)}\|^2_{L_2(\mathbb{T})} \le 1, \quad \sum_{|j|\le N} |x_j|^2 \le \delta^2, \quad x \in \mathcal{W}_2^r(\mathbb{T}).$$

The Lagrange function may be written in the following form

$$\mathcal{L}(x, \lambda_1, \lambda_2) = \sum_{|j|\le N} (-j^{2k} + \lambda_1 j^{2r} + \lambda_2)|x_j|^2 + \sum_{|j|>N} (-j^{2k} + \lambda_1 j^{2r})|x_j|^2.$$

Assume that

$$s^{2r} < \frac{1}{\delta^2} \le (s+1)^{2r},$$

$s < N$, and

$$\widehat{\lambda}_1 = \frac{(s+1)^{2k} - s^{2k}}{(s+1)^{2r} - s^{2r}} \ge \frac{1}{(N+1)^{2(r-k)}}.$$

Then for the same $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$ as in the previous case and any $x \in \mathcal{W}_2^r(\mathbb{T})$ we have

$$\mathcal{L}(x, \widehat{\lambda}_1, \widehat{\lambda}_2) \ge 0 = \mathcal{L}(\widehat{x}, \widehat{\lambda}_1, \widehat{\lambda}_2),$$

where $\widehat{x}$ is also the same as above.

Set

$$s_0 = \min\left\{ s \in \mathbb{N} : \frac{(s+1)^{2k} - s^{2k}}{(s+1)^{2r} - s^{2r}} \le \frac{1}{(N+1)^{2(r-k)}} \right\}. \tag{27}$$

Consider the line passing through the point (s_0^{2r}, s_0^{2k}) which is parallel to the line connected the origin and the point $((N+1)^{2r}, (N+1)^{2k})$. It has the form $y = \widehat{\lambda}_1 x + \widehat{\lambda}_2$, where

$$\widehat{\lambda}_1 = \frac{1}{(N+1)^{2(r-k)}}, \quad \widehat{\lambda}_2 = s_0^{2k} - \frac{s_0^{2r}}{(N+1)^{2(r-k)}}.$$

Now assume that $\delta^{-2} \geq s_0^{2r}$. Put $\widehat{x}_j = 0$, $j \neq s_0, N+1$, and $\widehat{x}_{s_0}, \widehat{x}_{N+1}$ define from the conditions

$$\|\widehat{x}^{(r)}\|^2_{L_2(\mathbb{T})} = 1, \quad \sum_{|j|\leq N} |\widehat{x}_j|^2 = \delta^2.$$

We put

$$\widehat{x}_{s_0} = \delta, \quad |\widehat{x}_{N+1}| = \frac{\sqrt{1-\delta^2 s_0^{2r}}}{(N+1)^r}.$$

The function

$$\widehat{x}(t) = \widehat{x}_{s_0} e^{is_0 t} + \widehat{x}_{N+1} e^{i(N+1)t}$$

is an admissible and consequently is extremal in the duality problem for the case when $\delta^{-2} \geq s_0^{2r}$.

Now we consider the extremal problem for finding an optimal method of recovery

$$\widehat{\lambda}_1 \sum_{j=-\infty}^{+\infty} j^{2r}|x_j|^2 + \widehat{\lambda}_2 \sum_{|j|\leq N} |x_j - \tilde{x}_j|^2 \to \min, \quad x \in \mathcal{W}_2^r(\mathbb{T}).$$

It may be rewritten in the following form

$$\sum_{|j|\leq N} (\widehat{\lambda}_1 j^{2r}|x_j|^2 + \widehat{\lambda}_2 |x_j - \tilde{x}_j|^2) + \widehat{\lambda}_1 \sum_{|j|>N} j^{2r}|x_j|^2 \to \min, \quad x \in \mathcal{W}_2^r(\mathbb{T}).$$

We can easily find the solution of this problem

$$x_j^0 = \begin{cases} \dfrac{\widehat{\lambda}_2}{\widehat{\lambda}_2 + j^{2r}\widehat{\lambda}_1}\tilde{x}_j, & |j| \leq N, \\ 0, & |j| > N. \end{cases}$$

It follows from Theorem 1 that the method

$$\widehat{m}(\tilde{x}) = \sum_{|j|\leq N} (ij)^k \frac{\widehat{\lambda}_2}{\widehat{\lambda}_2 + j^{2r}\widehat{\lambda}_1} \tilde{x}_j e^{ijt}$$

is optimal.

Lecture 11

Thus, for the problem

$$E_2^N(D^k, W_2^r(\mathbb{T}), \delta) = \inf_{m\colon \mathbb{C}^{2N+1}\to L_2(\mathbb{T})} \sup_{\substack{x\in W_2^r(\mathbb{T}),\ \tilde{x}=\{\tilde{x}_j\}_{|j|\le N} \\ \sum_{|j|\le N}|x_j-\tilde{x}_j|^2\le\delta^2}} \|x^{(k)} - m(\tilde{x})\|_{L_2(\mathbb{T})} \tag{28}$$

we obtain the following result.

Theorem 5. *Let $k, n, N \in \mathbb{N}$, $0 < k < n$, $\delta > 0$, and s_0 be defined by (27). Then for*

$$\frac{1}{(s+1)^r} \le \delta < \frac{1}{s^r}, \quad s = 1, 2, \ldots, s_0 - 1, \tag{29}$$

$$E_2^N(D^k, W_2^r(\mathbb{T}), \delta) = \sqrt{\delta^2 s^{2k} + (1-\delta^2 s^{2r})\frac{(s+1)^{2k} - s^{2k}}{(s+1)^{2r} - s^{2r}}}.$$

Moreover, the method

$$\widehat{m}(\tilde{x}) = \sum_{|j|<N} (ij)^k \left(1 + j^{2r}\frac{(s+1)^{2k} - s^{2k}}{s^{2k}(s+1)^{2r} - (s+1)^{2k}s^{2r}}\right)^{-1} \tilde{x}_j e^{ijt}$$

is optimal. For $\delta \ge 1$, $E_2^N(D^k, W_2^r(\mathbb{T}), \delta) = 1$ and the method $\widehat{m}(\tilde{x}) = 0$ is optimal. For $0 < \delta \le (s_0+1)^{-r}$,

$$E_2^N(D^k, W_2^r(\mathbb{T}), \delta) = \sqrt{\delta^2 s_0^{2k} + \frac{1-\delta^2 s_0^{2r}}{(N+1)^{2(r-k)}}}$$

and

$$\widehat{m}(\tilde{x}) = \sum_{|j|<N} (ij)^k \left(1 + \frac{j^{2r}}{s_0^{2k}(N+1)^{2(r-k)} - s_0^{2r}}\right)^{-1} \tilde{x}_j e^{ijt}$$

is an optimal method.

Now we wish to show that for δ satisfying condition (29) it is possible to construct an optimal method of recovery which uses, in general, less approximate values of Fourier coefficients. Set

$$N_s = \min\left\{ N \in \mathbb{N} : \frac{(s+1)^{2k} - s^{2k}}{(s+1)^{2r} - s^{2r}} > \frac{1}{(N+1)^{2(r-k)}} \right\}. \tag{30}$$

In view of definition of s_0, $N_s \le N$. It follows from Theorem 5 that

$$E_2^{N_s}(D^k, W_2^r(\mathbb{T}), \delta) = E_2^N(D^k, W_2^r(\mathbb{T}), \delta).$$

Denote by $\widehat{m}_1$ the optimal method of recovery obtained from Theorem 5 for $N = N_s$. We show that it is also optimal for the problem (28). We have

$$e_2^N(D^k, W_2^r(\mathbb{T}), \delta, \widehat{m}_1) = \sup_{\substack{x \in W_2^r(\mathbb{T}),\ \tilde{x}=\{\tilde{x}_j\}_{|j|\le N} \\ \sum_{|j|\le N} |x_j - \tilde{x}_j|^2 \le \delta^2}} \|x^{(k)} - \widehat{m}_1(\tilde{x})\|_{L_2(\mathbb{T})}$$
$$\le \sup_{\substack{x \in W_2^r(\mathbb{T}),\ \tilde{x}=\{\tilde{x}_j\}_{|j|\le N_s} \\ \sum_{|j|\le N_s} |x_j - \tilde{x}_j|^2 \le \delta^2}} \|x^{(k)} - \widehat{m}_1(\tilde{x})\|_{L_2(\mathbb{T})} = e_2^{N_s}(D^k, W_2^r(\mathbb{T}), \delta, \widehat{m}_1)$$
$$= E_2^{N_s}(D^k, W_2^r(\mathbb{T}), \delta) = E_2^N(D^k, W_2^r(\mathbb{T}), \delta).$$

Hence $\widehat{m}_1$ is optimal for the problem (28).

Now we can formulate a more precise version of Theorem 5.

Theorem 6. *Let $k, n, N \in \mathbb{N}$, $0 < k < n$, $\delta > 0$, s_0 be defined by* (27), *and N_s be defined by* (30). *Then for δ satisfying* (29)

$$E_2^N(D^k, W_2^r(\mathbb{T}), \delta) = \sqrt{\delta^2 s^{2k} + (1 - \delta^2 s^{2r}) \frac{(s+1)^{2k} - s^{2k}}{(s+1)^{2r} - s^{2r}}}.$$

Moreover, the method

$$\widehat{m}_1(\tilde{x}) = \sum_{|j|<N_s} (ij)^k \left(1 + j^{2r} \frac{(s+1)^{2k} - s^{2k}}{s^{2k}(s+1)^{2r} - (s+1)^{2k} s^{2r}}\right)^{-1} \tilde{x}_j e^{ijt}$$

is optimal. For $\delta \ge 1$, $E_2^N(D^k, W_2^r(\mathbb{T}), \delta) = 1$ and the method $\widehat{m}(\tilde{x}) = 0$ is optimal. For $0 < \delta \le (s_0+1)^{-r}$,

$$E_2^N(D^k, W_2^r(\mathbb{T}), \delta) = \sqrt{\delta^2 s_0^{2k} + \frac{1 - \delta^2 s_0^{2r}}{(N+1)^{2(r-k)}}}$$

and

$$\widehat{m}(\tilde{x}) = \sum_{|j|<N} (ij)^k \left(1 + \frac{j^{2r}}{s_0^{2k}(N+1)^{2(r-k)} - s_0^{2r}}\right)^{-1} \tilde{x}_j e^{ijt}$$

is an optimal method.

Let $0 < \delta < 1$ be fixed. Suppose that $s \in \mathbb{N}$ such that (29) is fulfilled. If we want to recover $x^{(k)}$ with the minimal error of optimal recovery and the minimal number of using inaccurate Furier coefficients, than this minimal number equals $2N_s(\delta)$.

Problems

Set

$$N_{kr}(\delta) = N_s, \quad \delta \in [(s+1)^{-r} \le \delta < s^{-r}).$$

1. Find the asymptotic of N_{kr} as $\delta \to 0$.
2. Find the asymptotic of $E_2^{N_{kr}}(D^k, W_2^r(\mathbb{T}), \delta) = E_2(D^k, W_2^r(\mathbb{T}), \delta)$ as $\delta \to 0$.

8. Optimal recovery of derivatives (continuous case)

We consider the analogous problem of recovery of derivatives for functions defined on $\mathbb{R}$. Namely, we want to recover $x^{(k)}$ by information about Fourier transform of x (which we denote by Fx) given with an error.

First we recall some facts about the Fourier transform. Let $x \in L_2(\mathbb{R})$. Then the Fourier transform of the function x is defined as follows

$$Fx(\tau) = \int_{\mathbb{R}} x(t)e^{-i\tau t}\, dt.$$

It follows from the Plancherel theorem that Fx can be considered as a function from $L_2(\mathbb{R})$, moreover,

$$\|x\|^2_{L_2(\mathbb{R})} = \frac{1}{2\pi}\|Fx\|^2_{L_2(\mathbb{R})}.$$

The inverse Fourier transform is given by the formula

$$x(t) = \frac{1}{2\pi}\int_{\mathbb{R}} Fx(\tau)e^{it\tau}\, d\tau.$$

We will need also the following well-known formula

$$Fx^{(k)}(\tau) = (i\tau)^k Fx(\tau).$$

Denote by $\mathcal{W}_2^r(\mathbb{R})$ the space of functions from $L_2(\mathbb{R})$ such that $x^{(r-1)}$ is locally absolute continuous on $\mathbb{R}$ and $x^{(r)} \in L_2(\mathbb{R})$. Let $W_2^r(\mathbb{R})$ be the class of functions from $\mathcal{W}_2^r(\mathbb{R})$ for which $\|x^{(r)}\|_{L_2(\mathbb{R})} \le 1$.

Lecture 12

We state the problem on optimal recovery of $x^{(k)}$, $0 < k < r$ on the class $W_2^r(\mathbb{R})$ in the $L_2(\mathbb{R})$-metric from the information about approximate values of Fourier transform Fx. Assume that for any $x \in W_2^r(\mathbb{R})$ we know a function $y \in L_2(\mathbb{R})$ such that

$$\|Fx - y\|_{L_2(\mathbb{R})} \le \delta.$$

Knowing y we want to recover $x^{(k)}$.

We define the error of optimal recovery as follows

$$E_2(D^k, W_2^r(\mathbb{R}), \delta) = \inf_{m\colon L_2(\mathbb{R})\to L_2(\mathbb{R})} \sup_{\substack{x\in W_2^r(\mathbb{R}),\ y\in L_2(\mathbb{R})\\ \|Fx-y\|_{L_2(\mathbb{R})}\le\delta}} \|x^{(k)} - m(y)\|_{L_2(\mathbb{R})}.$$

Any method for which the infimum is attained we call an optimal method of recovery.

Consider the duality problem

$$\|x^{(k)}\|^2_{L_2(\mathbb{T})} \to \max, \quad \|x^{(r)}\|^2_{L_2(\mathbb{R})} \le 1, \quad \|Fx\|^2_{L_2(\mathbb{R})} \le \delta^2, \quad x \in \mathcal{W}_2^r(\mathbb{R}).$$

Passing to Fourier transforms and using the Plancherel theorem, we may rewrite this problem in the form

(31)
$$\int_{\mathbb{R}} \tau^{2k} u(\tau)\, d\tau \to \max, \quad \int_{\mathbb{R}} \tau^{2r} u(\tau)\, d\tau \le 1, \quad 2\pi \int_{\mathbb{R}} u(\tau)\, d\tau \le \delta^2,$$
$$u \in L_1(\mathbb{R}), \quad u(\tau) \ge 0 \text{ almost everywhere on } \mathbb{R},$$

where $u(\tau) = (2\pi)^{-1}|Fx(\tau)|^2$. There is no existence of extremal function in this problem. Therefore, we consider the extension of this problem for measures

$$(32) \qquad \int_{\mathbb{R}} \tau^{2k}\, d\mu(\tau) \to \max, \quad \int_{\mathbb{R}} \tau^{2r}\, d\mu(\tau) \le 1, \quad 2\pi \int_{\mathbb{R}} d\mu(\tau) \le \delta^2.$$

The Lagrange function for this problem has the form

$$\mathcal{L}(\mu, \lambda_1, \lambda_2) = \int_{\mathbb{R}} (-\tau^{2k} + \lambda_1 \tau^{2r} + 2\pi\lambda_2)\, d\mu(\tau).$$

Consider the function

$$\begin{cases} y = \tau^{2k}, \\ x = \tau^{2r}. \end{cases}$$

We have $y = x^{k/r}$, $0 < k/r < 1$. Using the same arguments as above we want to find such $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$ that for all points of the curve $y = x^{k/r}$ the inequality $-y+\widehat{\lambda}_1 x+2\pi\widehat{\lambda}_2\geq 0$ will be fulfilled. Consider the tangent of this curve at some point $(\tau_0^{2r},\tau_0^{2k})$

$$y-\tau_0^{2k}=\frac{k}{r}\tau_0^{2k-2r}(x-\tau_0^{2r}).$$

Since the function $y=x^{k/r}$ is concave we have that for all points of this curve

$$-y+\frac{k}{r}\tau_0^{2k-2r}x+\tau_0^{2k}\frac{r-k}{r}\geq 0.$$

Set

$$\widehat{\lambda}_1=\frac{k}{r}\tau_0^{2k-2r},\quad \widehat{\lambda}_2=\frac{1}{2\pi}\tau_0^{2k}\frac{r-k}{r}.$$

Then for all τ

$$-\tau^{2k}+\widehat{\lambda}_1\tau^{2r}+2\pi\widehat{\lambda}_2\geq 0.$$

Hence for all μ, $\mathcal{L}(\mu,\widehat{\lambda}_1,\widehat{\lambda}_2)\geq 0$.

Now consider a measure concentrated at the point τ_0

$$d\widehat{\mu}(\tau)=A\delta(\tau-\tau_0).$$

Choose A and τ_0 from the conditions

$$\int_{\mathbb{R}}\tau^{2r}\,d\widehat{\mu}(\tau)=1,\quad 2\pi\int_{\mathbb{R}}d\widehat{\mu}(\tau)=\delta^2.$$

We have

$$A=\frac{\delta}{2\pi},\quad \tau_0=\left(\frac{2\pi}{\delta^2}\right)^{\frac{1}{2r}}.$$

Moreover, $\mathcal{L}(\widehat{\mu},\widehat{\lambda}_1,\widehat{\lambda}_2)=0$.

It follows from Theorem 2 that the value of the problem (32) coincides with the value of the problem

$$\int_{\mathbb{R}}\tau^{2k}\,d\mu(\tau)\to\max,\quad \int_{\mathbb{R}}(\widehat{\lambda}_1\tau^{2r}+2\pi\widehat{\lambda}_2)\,d\mu(\tau)\leq\widehat{\lambda}_1+\widehat{\lambda}_2\delta^2.$$

Since measures $A\delta(\tau-\tau_0)$ can be approximate by step functions, the value of (31) coincides with the value of the problem

$$\int_{\mathbb{R}}\tau^{2k}u(\tau)\,d\tau\to\max,\quad \int_{\mathbb{R}}(\widehat{\lambda}_1\tau^{2r}+2\pi\widehat{\lambda}_2)u(\tau)\,d\tau\leq\widehat{\lambda}_1+\widehat{\lambda}_2\delta^2,$$
$$u\in L_1(\mathbb{R}),\quad u(\tau)\geq 0 \text{ almost everywhere on } \mathbb{R},$$

Now it follows from Theorem 1 that it remains to find the solution of the extremal problem

$$\widehat{\lambda}_1\|x^{(r)}\|^2_{L_2(\mathbb{R})}+\widehat{\lambda}_2\|Fx-y\|^2_{L_2(\mathbb{T})}\to\min,\quad x\in\mathcal{W}_2^r(\mathbb{R}).$$

Passing to Fourier transforms and using the Plancherel theorem we obtain the following problem

$$\int_{\mathbb{R}}\left(\frac{\widehat{\lambda}_1}{2\pi}\tau^{2r}|Fx(\tau)|^2+\widehat{\lambda}_2|Fx(\tau)-y(\tau)|^2\right)d\tau\to\min,\quad x\in\mathcal{W}_2^r(\mathbb{R}).$$

It can be easily verified that the solution of this problem is the function x_0 such that

$$Fx_0(\tau)=\left(1+\frac{\tau^{2r}\widehat{\lambda}_1}{2\pi\widehat{\lambda}_2}\right)^{-1}y=\left(1+\frac{\delta^2}{2\pi}\frac{k}{r-k}\tau^{2r}\right)^{-1}y.$$

Thus, we prove

Theorem 7. *Let $k,r\in\mathbb{N}$, $0<k<n$, and $\delta>0$. Then*

$$E_2(D^k,W_2^r(\mathbb{R}),\delta)=\left(\frac{\delta}{\sqrt{2\pi}}\right)^{1-k/r}$$

and the method

$$\widehat{m}(y)=\int_{\mathbb{R}}(i\tau)^k\left(1+\frac{\delta^2}{2\pi}\frac{k}{r-k}\tau^{2r}\right)^{-1}ye^{it\tau}\,d\tau$$

is optimal.

Lecture 13

It follows from Theorems 1 and 7 that

$$\sup_{\substack{x\in W_2^r(\mathbb{R})\\ \|Fx\|_{L_2(\mathbb{R})}\le\delta}} \|x^{(k)}\|_{L_2(\mathbb{R})} = \left(\frac{\delta}{\sqrt{2\pi}}\right)^{1-k/r}.$$

It means that for all $x \in W_2^r(\mathbb{R})$ such that $\|Fx\|_{L_2(\mathbb{R})} \le \delta$

$$\|x^{(k)}\|_{L_2(\mathbb{R})} \le \left(\frac{\delta}{\sqrt{2\pi}}\right)^{1-k/r}. \tag{33}$$

Let $f \in \mathcal{W}_2^r(\mathbb{R})$ and $f \ne 0$. Put

$$x = \frac{f}{\|f^{(r)}\|_{L_2(\mathbb{R})}}, \quad \delta = \|Fx\|_{L_2(\mathbb{R})} = \frac{\|Ff\|_{L_2(\mathbb{R})}}{\|f^{(r)}\|_{L_2(\mathbb{R})}}.$$

Substituting x to (33) we obtain

$$\frac{\|f^{(k)}\|_{L_2(\mathbb{R})}}{\|f^{(r)}\|_{L_2(\mathbb{R})}} \le \left(\frac{1}{2\pi}\right)^{\frac{r-k}{2r}} \left(\frac{\|Ff\|_{L_2(\mathbb{R})}}{\|f^{(r)}\|_{L_2(\mathbb{R})}}\right)^{\frac{r-k}{r}}.$$

Thus we obtain the following inequality

$$\|f^{(k)}\|_{L_2(\mathbb{R})} \le \left(\frac{1}{2\pi}\right)^{\frac{r-k}{2r}} \|Ff\|_{L_2(\mathbb{R})}^{1-k/r} \|f^{(r)}\|_{L_2(\mathbb{R})}^{k/r}. \tag{34}$$

This inequality is exact. It means that we cannot replace the number $(2\pi)^{-(r-k)/(2r)}$ by any smaller number.

In view of the equality

$$\|Ff\|_{L_2(\mathbb{R})}^2 = 2\pi\|f\|_{L_2(\mathbb{R})}^2$$

it follows from (34) that

$$\|f^{(k)}\|_{L_2(\mathbb{R})} \le \|f\|_{L_2(\mathbb{R})}^{1-k/r} \|f^{(r)}\|_{L_2(\mathbb{R})}^{k/r}. \tag{35}$$

The last inequality is known as the Hardy–Littlewood–Pólya inequality. It is the one from a big set of the so-called Landau–Kolmogorov type inequalities for derivatives.

9. Landau–Kolmogorov inequalities for derivatives and optimal recovery

Exact inequalities for derivatives have been attracting the attention of many mathematicians for many years. The first result in this field was obtained by E. Landau in 1913 who proved that for all functions

$x \in L_\infty(\mathbb{R}_+)$ with the first derivative locally absolutely continuous on $\mathbb{R}_+$ and $x'' \in L_\infty(\mathbb{R}_+)$ the following exact inequality

$$\|x'\|_{L_\infty(\mathbb{R}_+)} \le 2\|x\|^{1/2}_{L_\infty(\mathbb{R}_+)}\|x''\|^{1/2}_{L_\infty(\mathbb{R}_+)}$$

holds. Then in 1914 Hadamard proved the exact inequality

$$\|x'\|_{L_\infty(\mathbb{R})} \le \sqrt{2}\|x\|^{1/2}_{L_\infty(\mathbb{R})}\|x''\|^{1/2}_{L_\infty(\mathbb{R})}.$$

The first general result was obtained by Hardy, Littlewood, and Pólya. In 1934 they proved inequality (35).

Probably the most remarkable result was obtained by Kolmogorov in 1939 who proved that

$$\|x^{(k)}\|_{L_\infty(\mathbb{R})} \le \frac{K_{r-k}}{K_r^{1-\frac{k}{r}}}\|x\|^{1-k/r}_{L_\infty(\mathbb{R})}\|x^{(r)}\|^{k/r}_{L_\infty(\mathbb{R})},$$

where

$$K_m = \frac{4}{\pi}\sum_{s=0}^{\infty}\frac{(-1)^{s(m+1)}}{(2s+1)^{m+1}}$$

are the Favard constants.

Let $\mathcal{W}_s^r(T)$ be the set of all functions x with the $(r-1)$st derivative locally absolutely continuous on $T = \mathbb{R}$ or $\mathbb{R}_+$ and $x^{(r)} \in L_p(T)$. The general problem of Landau–Kolmogorov type exact inequalities may be formulated as follows: find a minimal constant $K = K(k, r, p, q, s)$ such that for all functions $x \in \mathcal{W}_s^r(T) \cap L_q(T)$ the inequality

$$\|x^{(k)}\|_{L_p(T)} \le K\|x\|^{\alpha}_{L_q(T)}\|x^{(r)}\|^{\beta}_{L_s(T)} \tag{36}$$

holds, where $0 \le k < r$, $1 \le p, q, s \le \infty$.

If there exists a constant K that for all $x \in \mathcal{W}_s^r(T) \cap L_q(T)$ inequality (36) is fulfilled, then $\alpha + \beta = 1$. Indeed, let $x \ne 0$ be a function from $\mathcal{W}_s^r(T) \cap L_q(T)$. Consider the function λx, $\lambda > 0$. Substituting this function in (36), we obtain

$$\lambda\|x^{(k)}\|_{L_p(T)} \le \lambda^{\alpha+\beta}K\|x\|^{\alpha}_{L_q(T)}\|x^{(r)}\|^{\beta}_{L_s(T)}.$$

The only case to have such inequality for all $\lambda > 0$ is the case when $\alpha + \beta = 1$.

Now consider the function $x(\lambda t)$. We have

$$\|x(\lambda t)\|_{L_p(T)} = \left(\int_{\mathbb{R}}|x(\lambda t)|^p\,dt\right)^{1/p} = \left(\frac{1}{\lambda}\int_{\mathbb{R}}|x(\tau)|^p\,d\tau\right)^{1/p}$$
$$= \lambda^{-1/p}\|x\|_{L_p(T)}.$$

Substituting the function $x(\lambda t)$ in (36), we obtain

$$\lambda^{k-1/p}\|x^{(k)}\|_{L_p(T)} \le K\lambda^{-(1-\beta)/q}\|x\|^{1-\beta}_{L_q(T)}\lambda^{(r-1/s)\beta}\|x^{(r)}\|^{\beta}_{L_s(T)}.$$

Thus we have

$$k - 1/p = -(1-\beta)/q + (r - 1/s)\beta.$$

Hence

$$\beta = \frac{k + 1/q - 1/p}{r + 1/q - 1/s}.$$

We proved that if there exists a constant K that for all $x \in \mathcal{W}^r_s(T) \cap L_q(T)$ inequality (36) is fulfilled, then this inequality should have the following form

$$\|x^{(k)}\|_{L_p(T)} \le K\|x\|^{\frac{r-k+1/p-1/s}{r+1/q-1/s}}_{L_q(T)}\|x^{(r)}\|^{\frac{k+1/q-1/p}{r+1/q-1/s}}_{L_s(T)}. \tag{37}$$

Lecture 14

Proposition 2. *If K is the exact constant in* (37), *then for all $\delta > 0$*

$$\sup_{\substack{x\in\mathcal{W}_s^r(T)\cap L_q(T)\\ \|x\|_{L_q(T)}\le\delta\\ \|x^{(r)}\|_{L_s(T)}\le 1}} \|x^{(k)}\|_{L_p(T)} = K\delta^{\frac{r-k+1/p-1/s}{r+1/q-1/s}}.$$

Proof. Since K is the exact constant in (37), for any $\varepsilon > 0$ there exists a function $x_\varepsilon \in \mathcal{W}_s^r(T)\cap L_q(T)$, $x \neq 0$, such that

$$\|x_\varepsilon^{(k)}\|_{L_p(T)} = (K-\varepsilon)\|x_\varepsilon\|_{L_q(T)}^{\frac{r-k+1/p-1/s}{r+1/q-1/s}}\|x_\varepsilon^{(r)}\|_{L_s(T)}^{\frac{k+1/q-1/p}{r+1/q-1/s}}.$$

For the function $f_\varepsilon(t) = Ax_\varepsilon(\lambda t)$, $A, \lambda > 0$, we have

$$\|f_\varepsilon^{(r)}\|_{L_s(T)} = A\lambda^{r-1/s}\|x_\varepsilon^{(r)}\|_{L_s(T)}, \quad \|f_\varepsilon\|_{L_q(T)} = A\lambda^{-1/q}\|x_\varepsilon\|_{L_q(T)}.$$

Putting

$$\lambda = \left(\frac{\|x_\varepsilon\|_{L_q(T)}}{\delta\|x_\varepsilon^{(r)}\|_{L_s(T)}}\right)^{\frac{1}{r+1/q-1/s}}, \quad A = \frac{1}{\lambda^{r-1/s}\|x_\varepsilon^{(r)}\|_{L_s(T)}},$$

we obtain

$$\|f_\varepsilon^{(r)}\|_{L_s(T)} = 1, \quad \|f_\varepsilon\|_{L_q(T)} = \delta.$$

Consequently,

$$\sup_{\substack{x\in\mathcal{W}_s^r(T)\cap L_q(T)\\ \|x\|_{L_q(T)}\le\delta\\ \|x^{(r)}\|_{L_s(T)}\le 1}} \|x^{(k)}\|_{L_p(T)} \ge \|f_\varepsilon^{(k)}\|_{L_s(T)} = (K-\varepsilon)\delta^{\frac{r-k+1/p-1/s}{r+1/q-1/s}}.$$

Since ε is an arbitrary positive number we have

$$\sup_{\substack{x\in\mathcal{W}_s^r(T)\cap L_q(T)\\ \|x\|_{L_q(T)}\le\delta\\ \|x^{(r)}\|_{L_s(T)}\le 1}} \|x^{(k)}\|_{L_p(T)} \ge K\delta^{\frac{r-k+1/p-1/s}{r+1/q-1/s}}.$$

The upper bound follows immediately from (37). □

Corollary 1.

$$K = \sup_{\substack{x\in\mathcal{W}_s^r(T)\cap L_q(T)\\ \|x\|_{L_q(T)}\le 1\\ \|x^{(r)}\|_{L_s(T)}\le 1}} \|x^{(k)}\|_{L_p(T)}$$

is the exact constant in (37).

Now we establish the connection of optimal recovery problems with the exact constants in Landau-Kolmogorov inequalities for derivatives. Consider the problem of optimal recovery of $x^{(k)}$, $x \in W_s^r(T) \cap L_q(T)$, in $L_p(T)$-metric on the basis of inaccurate information about x, where $W_s^r(T)$ is the set of functions from $\mathcal{W}_s^r(T)$ for which $\|x^{(r)}\|_{L_s(T)} \le 1$. We assume that for all $x \in W_s^r(T) \cap L_q(T)$ we know a function $y \in L_q(T)$ such that $\|x - y\|_{L_q(T)} \le \delta$. Knowing y we want to recover $x^{(k)}$ in an optimal way. In this case the error of optimal recovery is defined as follows

$$E_q^*(D^k, W_s^r(T) \cap L_q(T), \delta) = \inf_{m\colon L_q(T) \to L_p(T)} \sup_{x \in W_s^r(T) \cap L_q(T)} \sup_{\substack{y \in L_q(T) \\ \|x-y\|_{L_q(T)} \le \delta}} \|x^{(k)} - m(y)\|_{L_p(T)}.$$

It follows from Lemma 2 that

$$E_q^*(D^k, W_s^r(T) \cap L_q(T), \delta) \ge \sup_{\substack{x \in \mathcal{W}_s^r(T) \cap L_q(T) \\ \|x\|_{L_q(T)} \le \delta \\ \|x^{(r)}\|_{L_s(T)} \le 1}} \|x^{(k)}\|_{L_p(T)}.$$

Thus we obtain the following result.

Theorem 8. *If K is the exact constant in equality* (37), *then for all* $\delta > 0$

$$E_q^*(D^k, W_s^r(T) \cap L_q(T), \delta) \ge K \delta^{\frac{r-k+1/p-1/s}{r+1/q-1/s}}.$$

10. Inequality for derivatives with Fourier transform

In (34) we obtain the exact inequality where we estimate the k-th derivative by the r-th derivative and the Fourier transform of function. Consider the following general problem. Let $\mathcal{F}_{sq}^r$ denote the space of functions $x \in \mathcal{W}_s^r(\mathbb{R})$ for which $Fx \in L_q(\mathbb{R})$. The problem is to find a minimal constant $K_F = K_F(k, r, p, q, s)$ such that for all functions $x \in \mathcal{F}_{sq}^r$ the inequality

$$\|x^{(k)}\|_{L_p(\mathbb{R})} \le K_F \|Fx\|_{L_q(\mathbb{R})}^{\alpha} \|x^{(r)}\|_{L_s(\mathbb{R})}^{\beta} \tag{38}$$

holds, where $0 \le k < r$, $1 \le p, q, s \le \infty$.

The same arguments as above show that $\alpha + \beta = 1$. Now consider the function $x_\lambda(t) = x(\lambda t)$. We have

$$Fx_\lambda(\tau) = \int_{\mathbb{R}} x(\lambda t) e^{-i\tau t}\, d\tau = \frac{1}{\lambda} \int_{\mathbb{R}} x(u) e^{-iu\tau/\lambda}\, du = \frac{1}{\lambda} Fx\left(\frac{\tau}{\lambda}\right).$$

Substituting the function $x(\lambda t)$ in (38), we obtain

$$\lambda^{k-1/p} \|x^{(k)}\|_{L_p(\mathbb{R})} \le K_F \lambda^{-(1-\beta)(q-1)/q} \|Fx\|_{L_q(\mathbb{R})}^{1-\beta} \lambda^{(r-1/s)\beta} \|x^{(r)}\|_{L_s(\mathbb{R})}^{\beta}.$$

Thus we have

$$k - 1/p = -(1-\beta)(q-1)/q + (r - 1/s)\beta.$$

Consequently,

$$\beta = \frac{k + 1/q' - 1/p}{r + 1/q' - 1/s},$$

where q' is defined as follows

$$\frac{1}{q} + \frac{1}{q'} = 1.$$

We proved that if there exists a constant K_F that for all $x \in \mathcal{F}^r_{sq}$ inequality (38) is fulfilled, then this inequality should have the following form

$$\|x^{(k)}\|_{L_p(\mathbb{R})} \le K_F \|Fx\|_{L_q(\mathbb{R})}^{\frac{r-k+1/p-1/s}{r+1/q'-1/s}} \|x^{(r)}\|_{L_s(\mathbb{R})}^{\frac{k+1/q'-1/p}{r+1/q'-1/s}}. \tag{39}$$

Lecture 15

Similarly to Proposition 2 we obtain

Proposition 3. *If K_F is the exact constant in* (39)*, then for all* $\delta > 0$

$$\sup_{\substack{x\in\mathcal{F}^r_{sq}\\ \|Fx\|_{L_q(\mathbb{R})}\le\delta\\ \|x^{(r)}\|_{L_s(\mathbb{R})}\le 1}} \|x^{(k)}\|_{L_p(\mathbb{R})} = K_F\delta^{\frac{r-k+1/p-1/s}{r+1/q'-1/s}}.$$

Corollary 2.

$$K_F = \sup_{\substack{x\in\mathcal{F}^r_{sq}\\ \|Fx\|_{L_q(\mathbb{R})}\le 1\\ \|x^{(r)}\|_{L_s(\mathbb{R})}\le 1}} \|x^{(k)}\|_{L_p(\mathbb{R})}$$

is the exact constant in (39)*.*

Now we state the problem of optimal recovery of $x^{(k)}$, $x \in F^r_{sq}$, in $L_p(T)$-metric on the basis of inaccurate information about Fx, where $F^r_{sq} = \mathcal{F}^r_{sq}\cap W^r_s(\mathbb{R})$. We assume that for all $x \in F^r_{sq}$ we know a function $y \in L_q(T)$ such that $\|Fx-y\|_{L_q(T)} \le \delta$. Knowing y we want to recover $x^{(k)}$ in an optimal way. In this case the error of optimal recovery is defined as follows

$$E_q(D^k, F^r_{sq}, \delta) = \inf_{m\colon L_q(T)\to L_p(\mathbb{R})} \sup_{x\in F^r_{sq}} \sup_{\substack{y\in L_q(T)\\ \|Fx-y\|_{L_q(\mathbb{R})}\le\delta}} \|x^{(k)} - m(y)\|_{L_p(\mathbb{R})}.$$

It follows from Lemma 2 that

$$E_q(D^k, F^r_{sq}, \delta) \ge \sup_{\substack{x\in\mathcal{F}^r_{sq}\\ \|Fx\|_{L_q(\mathbb{R})}\le\delta\\ \|x^{(r)}\|_{L_s(\mathbb{R})}\le 1}} \|x^{(k)}\|_{L_p(\mathbb{R})}.$$

The analog of Theorem 8 is

Theorem 9. *If K_F is the exact constant in equality* (39)*, then for all* $\delta > 0$

$$E_q(D^k, F^r_{sq}, \delta) \ge K_F\delta^{\frac{r-k+1/p-1/s}{r+1/q'-1/s}}.$$

It follows from (34) (since in this inequality the constant is exact) that

$$K_F(k, r, 2, 2, 2) = \left(\frac{1}{2\pi}\right)^{\frac{r-k}{2r}}.$$

Now we find the exact constant $K_F(k, r, 2, q, 2)$ for $2 < q < \infty$.

Theorem 10. *Let $n \in \mathbb{N}$, $0 \le k < r$, and $2 < q < \infty$. Then*

$$K_F(k,r,2,q,2)$$
$$= \sqrt{\frac{r+1/2-1/q}{k+1/2-1/q}}\left(\frac{\sqrt{k+1/2-1/q}\,B^{1/2-1/q}}{\sqrt{2\pi}(r-k)^{1-1/q}}\right)^{\frac{r-k}{r+1/2-1/q}},$$

where

$$B = B\left(\frac{k+1/2-1/q}{(r-k)(1-2/q)}, 2\frac{1-1/q}{1-2/q}\right) \tag{40}$$

and

$$B(a,b) = \int_0^1 t^{a-1}(1-x)^{b-1}\,dx$$

is the Euler beta function.

Consider the extremal problem

$$\|x^{(k)}\|^2_{L_2(\mathbb{R})} \to \max, \quad \|Fx\|^2_{L_q(\mathbb{R})} \le 1, \quad \|x^{(r)}\|^2_{L_2(\mathbb{R})} \le 1.$$

This problem can be rewritten in terms of the Fourier transforms as

$$\int_{\mathbb{R}} t^{2k}u(t)\,dt \to \max, \quad \int_{\mathbb{R}} u^{q/2}(t)\,dt \le \left(\frac{1}{2\pi}\right)^{q/2}, \quad \int_{\mathbb{R}} t^{2r}u(t)\,dt \le 1,\ u(t) \ge 0, \tag{41}$$

where $u = (2\pi)^{-1}|Fx|^2$. For this problem the Lagrange function has the form

$$\mathcal{L}(u,\lambda_1,\lambda_2) = \int_{\mathbb{R}} (-t^{2k}u(t) + \lambda_1 u^{p/2}(t) + \lambda_2 t^{2r}u(t))\,dt.$$

It follows from Theorem 2 that if we find a function $\widehat{u}$ admissible in (41) and Lagrange multipliers $\widehat{\lambda}_1, \widehat{\lambda}_2 \ge 0$ such that

$$(a) \quad \min_{u(t)\ge 0} \mathcal{L}(u,\widehat{\lambda}_1,\widehat{\lambda}_2) = \mathcal{L}(\widehat{u},\widehat{\lambda}_1,\widehat{\lambda}_2),$$
$$(b) \quad \widehat{\lambda}_1\left(\int_{\mathbb{R}} u(t)^{q/2}\,dt - \frac{1}{2\pi}\right) = 0,$$
$$(c) \quad \widehat{\lambda}_2\left(\int_{\mathbb{R}} t^{2r}u(t)\,dt - 1\right) = 0,$$

then $\widehat{u}$ will be a solution of problem (41). Set $\widehat{\lambda}_2 = \sigma^{-2(r-k)}$, where parameter $\sigma > 0$ will be defined later. Since for any fixed t and $\sigma \ge t$ the function

$$f(x) = -t^{2k}x + \widehat{\lambda}_1 x^{q/2} + \frac{t^{2r}}{\sigma^{2(r-k)}}x$$

attains its minimum at the point

$$\widehat{x} = \left(\frac{2}{q\widehat{\lambda}_1}\left(t^{2k} - \frac{t^{2r}}{\sigma^{2(r-k)}}\right)\right)^{\frac{1}{q/2-1}},$$

we have

$$-t^{2k}u(t) + \widehat{\lambda}_1 u^{q/2}(t) + \frac{t^{2r}}{\sigma^{2(r-k)}}u(t) \geq -t^{2k}\widehat{u}(t) + \lambda_1 \widehat{u}^{q/2}(t) + \frac{t^{2r}}{\sigma^{2(r-k)}}\widehat{u}(t)$$

for all $u(t) \geq 0$ and any $\widehat{\lambda}_1 > 0$, where

$$\widehat{u}(t) = \begin{cases} \left(\frac{2}{q\widehat{\lambda}_1}\left(t^{2k} - \frac{t^{2r}}{\sigma^{2(r-k)}}\right)\right)^{\frac{1}{q/2-1}}, & |t| \leq \sigma, \\ 0, & |t| > \sigma. \end{cases}$$

Thus, condition (a) is satisfied. We take σ and $\widehat{\lambda}_1$ such that conditions (b) and (c) are satisfied:

$$\mu^{q/2}\int_{-\sigma}^{\sigma}\left(t^{2k} - \frac{t^{2r}}{\sigma^{2(r-k)}}\right)^{\frac{q/2}{q/2-1}} dt = \left(\frac{1}{2\pi}\right)^{q/2},$$

$$\mu\int_{-\sigma}^{\sigma} t^{2r}\left(t^{2k} - \frac{t^{2r}}{\sigma^{2(r-k)}}\right)^{\frac{1}{q/2-1}} dt = 1,$$

where

$$\mu = \left(\frac{2}{q\widehat{\lambda}_1}\right)^{\frac{1}{q/2-1}}.$$

Making the change of variable $t = \sigma y$, we obtain

$$2\mu^{q/2}\sigma^{\frac{qk}{q/2-1}+1}\int_0^1 y^{\frac{qk}{q/2-1}}\left(1 - y^{2(r-k)}\right)^{\frac{q/2}{q/2-1}} dy = \left(\frac{1}{2\pi}\right)^{q/2},$$

$$2\mu\sigma^{\frac{2k}{q/2-1}+2r+1}\int_0^1 y^{\frac{2k+r(q-2)}{q/2-1}}\left(1 - y^{2(r-k)}\right)^{\frac{1}{q/2-1}} dy = 1.$$

Now putting

$$y = \tau^{\frac{1}{2(r-k)}},$$

we obtain

$$\mu^{q/2}\sigma^{\frac{qk}{q/2-1}+1}\frac{1}{r-k}\int_0^1 \tau^{\frac{k+1/2-1/q}{(r-k)(1-2/q)}-1}(1-\tau)^{\frac{q/2}{q/2-1}} d\tau = \left(\frac{1}{2\pi}\right)^{q/2},$$

$$\mu\sigma^{\frac{2k}{q/2-1}+2r+1}\frac{1}{r-k}\int_0^1 \tau^{\frac{k+1/2-1/q}{(r-k)(1-2/q)}}(1-\tau)^{\frac{1}{q/2-1}} d\tau = 1.$$

Expressing the resulting integrals via the value of beta function B defined in (40) and using the property of beta function

$$B(a+1,b)=\frac{a}{b}B(a,b+1),$$

we obtain

$$\mu^{q/2}\sigma^{\frac{qk}{q/2-1}+1}\frac{B}{r-k}=\left(\frac{1}{2\pi}\right)^{q/2},$$

$$\mu\sigma^{\frac{2k}{q/2-1}+2r+1}\frac{(k+1/2-1/q)B}{(r-k)^2}=1.$$

Hence

$$\mu=\frac{(r-k)^2}{(k+1/2-1/q)B}\sigma^{-\frac{2k}{q/2-1}-2r-1} \tag{42}$$

and

$$\sigma=\left(\frac{\sqrt{2\pi}(r-k)^{1-1/q}}{(k+1/2-1/q)^{1/2}B^{1/2-1/q}}\right)^{\frac{1}{r+1/2-1/q}}. \tag{43}$$

Taking into account (42), we have

$$\int_{\mathbb{R}}t^{2k}\widehat{u}(t)\,dt=\frac{r+1/2-1/q}{k+1/2-1/p}\sigma^{-2(r-k)}.$$

Substituting there the value σ given by (43), we obtain that for all $2<q<\infty$

$$\sup_{\substack{x\in\mathcal{F}_{2q}^r\\ \|Fx\|_{L_q(\mathbb{R})}\le 1\\ \|x^{(r)}\|_{L_2(\mathbb{R})}\le 1}}\|x^{(k)}\|_{L_2(\mathbb{R})}$$

$$=\sqrt{\frac{r+1/2-1/q}{k+1/2-1/q}}\left(\frac{\sqrt{k+1/2-1/q}B^{1/2-1/q}}{\sqrt{2\pi}(n-k)^{1-1/q}}\right)^{\frac{r-k}{r+1/2-1/q}}.$$

Now the theorem follows from Corollary 2.

Lecture 16

11. Optimal recovery of derivatives from Fourier transforms given on a finite interval

Let us return to the problem of optimal recovery of the k-th derivative of functions from $W_2^r(\mathbb{R})$ on the basis of inaccurate information about their Fourier transforms. But now we will consider the case when the Fourier transform Fx is given on a finite interval $\Delta_\sigma = (-\sigma, \sigma)$, $\sigma > 0$.

We assume that for any function $x \in W_2^r(\mathbb{R})$ we know $y \in L_2(\Delta_\sigma)$ such that

$$\|Fx - y\|_{L_2(\Delta_\sigma)} \le \delta.$$

The error of optimal recovery is defined as follows

$$E_2^\sigma(D^k, W_2^r(\mathbb{R}), \delta) = \inf_{m\colon L_2(\Delta_\sigma)\to L_2(\mathbb{R})} \sup_{\substack{x\in W_2^r(\mathbb{R}),\ y\in L_2(\Delta_\sigma)\\ \|Fx-y\|_{L_2(\Delta_\sigma)}\le\delta}} \|x^{(k)} - m(y)\|_{L_2(\mathbb{R})}.$$

In this case the dual problem has the form

$$(44)\quad \|x^{(k)}\|^2_{L_2(\mathbb{T})} \to \max, \quad \|x^{(r)}\|^2_{L_2(\mathbb{R})} \le 1, \quad \|Fx\|^2_{L_2(\Delta_\sigma)} \le \delta^2, \quad x \in \mathcal{W}_2^r(\mathbb{R}).$$

Passing to Fourier transforms and using the Plancherel theorem, we may rewrite this problem in the form

(45)

$$\int_{\mathbb{R}} \tau^{2k} u(\tau)\, d\tau \to \max, \quad \int_{\mathbb{R}} \tau^{2r} u(\tau)\, d\tau \le 1, \quad 2\pi \int_{\Delta_\sigma} u(\tau)\, d\tau \le \delta^2,$$
$$u \in L_1(\mathbb{R}), \quad u(\tau) \ge 0 \text{ almost everywhere on } \mathbb{R},$$

where $u(\tau) = (2\pi)^{-1}|Fx(\tau)|^2$. Since there is no existence we, again consider the extension of this problem for measures

$$(46)\quad \int_{\mathbb{R}} \tau^{2k}\, d\mu(\tau) \to \max, \quad \int_{\mathbb{R}} \tau^{2r}\, d\mu(\tau) \le 1, \quad 2\pi \int_{\Delta_\sigma} d\mu(\tau) \le \delta^2.$$

The Lagrange function for this problem has the form

$$\mathcal{L}(\mu, \lambda_1, \lambda_2) = \int_{\mathbb{R}} (-\tau^{2k} + \lambda_1 \tau^{2r} + 2\pi\lambda_2 \chi_\sigma(t))\, d\mu(\tau),$$

where

$$\chi_\sigma(t) = \begin{cases} 1, & t \in (-\sigma, \sigma),\\ 0, & t \notin (-\sigma, \sigma). \end{cases}$$

Consider the function

$$\begin{cases} y = \tau^{2k}, \\ x = \tau^{2r}. \end{cases}$$

We have $y = x^{k/r}$, $0 < k/r < 1$. Consider the tangent of this curve at some point $(\tau_0^{2r}, \tau_0^{2k})$

$$y - \tau_0^{2k} = \frac{k}{r}\tau_0^{2k-2r}(x - \tau_0^{2r}).$$

Since the function $y = x^{k/r}$ is concave we have that for all points of this curve

$$-y + \frac{k}{r}\tau_0^{2k-2r}x + \tau_0^{2k}\frac{r-k}{r} \geq 0.$$

Set

$$\widehat{\lambda}_1 = \frac{k}{r}\tau_0^{2k-2r}, \quad \widehat{\lambda}_2 = \frac{1}{2\pi}\tau_0^{2k}\frac{r-k}{r}.$$

Then for all τ

$$-\tau^{2k} + \widehat{\lambda}_1\tau^{2r} + 2\pi\widehat{\lambda}_2 \geq 0.$$

Now let us find $\widehat{\sigma}$ such that for all $\tau \geq \widehat{\sigma}$

$$-\tau^{2k} + \widehat{\lambda}_1\tau^{2r} \geq 0.$$

It can be easily obtained that

$$\widehat{\sigma} = \widehat{\lambda}_1^{-\frac{1}{2(r-k)}} = \left(\frac{r}{k}\right)^{\frac{1}{2(r-k)}}\tau_0.$$

Assume that $\sigma \geq \widehat{\sigma}$. Then for all μ

$$\mathcal{L}(\mu, \widehat{\lambda}_1, \widehat{\lambda}_2) = \int_{\Delta_\sigma} (-\tau^{2k} + \widehat{\lambda}_1\tau^{2r} + 2\pi\widehat{\lambda}_2)\, d\mu(\tau) + \int_{\mathbb{R}\setminus\Delta_\sigma} (-\tau^{2k} + \widehat{\lambda}_1\tau^{2r})\, d\mu(\tau) \geq 0.$$

Now consider a measure concentrated at the point τ_0

$$d\widehat{\mu}(\tau) = A\delta(\tau - \tau_0).$$

Choose A and τ_0 from the conditions

$$\int_{\mathbb{R}} \tau^{2r}\, d\widehat{\mu}(\tau) = 1, \quad 2\pi\int_{\Delta_\sigma} d\widehat{\mu}(\tau) = \delta^2.$$

We have

$$A = \frac{\delta^2}{2\pi}, \quad \tau_0 = \left(\frac{2\pi}{\delta^2}\right)^{\frac{1}{2r}}.$$

Moreover, $\mathcal{L}(\widehat{\mu}, \widehat{\lambda}_1, \widehat{\lambda}_2) = 0$.

Thus, it follows from Theorem 2 that for the case

$$\sigma \geq \widehat{\sigma} = \left(\frac{r}{k}\right)^{\frac{1}{2(r-k)}} \tau_0 = \left(\frac{r}{k}\right)^{\frac{1}{2(r-k)}} \left(\frac{2\pi}{\delta^2}\right)^{\frac{1}{2r}}$$

we solved the extremal problem (46).

Lecture 17

Now we consider the case when $\sigma < \widehat{\sigma}$. The line $y = \sigma^{2(k-r)}x$ passes through the points $(0,0)$ and $(\sigma^{2k}, \sigma^{2r})$. Let us find a point $\widehat{\tau}$ such that the tangent of the curve $y = x^{k/r}$ at the point $\widehat{\tau}^{2r}$ is parallel to the line $y = \sigma^{2(k-r)}x$. We have

$$\frac{k}{r}(\widehat{\tau}^{2r})^{k/r-1} = \sigma^{2(k-r)}.$$

Hence

$$\widehat{\tau} = \left(\frac{k}{r}\right)^{\frac{1}{2(r-k)}} \sigma.$$

The equation of the tangent has the form

$$y = \widehat{\lambda}_1 x + 2\pi\widehat{\lambda}_2, \tag{47}$$

where

$$\widehat{\lambda}_1 = \sigma^{2(k-r)}, \quad \widehat{\lambda}_2 = \frac{1}{2\pi}\frac{r-k}{r}\left(\frac{k}{r}\right)^{\frac{k}{r-k}} \sigma^{2k}.$$

Since the function $y = x^{k/r}$ is concave and the line (47) is a tangent, we have that for all points of this curve

$$-y + \widehat{\lambda}_1 x + 2\pi\widehat{\lambda}_2 \ge 0.$$

Moreover, for all $t \ge \sigma$, $-t^{2k} + \widehat{\lambda}_1 t^{2r} \ge 0$. Thus for all μ

$$\mathcal{L}(\mu, \widehat{\lambda}_1, \widehat{\lambda}_2) = \int_{\Delta_\sigma} (-\tau^{2k} + \widehat{\lambda}_1\tau^{2r} + 2\pi\widehat{\lambda}_2)\, d\mu(\tau) + \int_{\mathbb{R}\setminus\Delta_\sigma} (-\tau^{2k} + \widehat{\lambda}_1\tau^{2r})\, d\mu(\tau) \ge 0.$$

Now we put

$$d\widehat{\mu}(t) = A\delta(t - \widehat{\tau}) + B\delta(t - \sigma),$$

where $A > 0$ and $B > 0$ are defined from the conditions

$$\int_{\mathbb{R}} \tau^{2r}\, d\widehat{\mu}(\tau) = 1, \quad 2\pi\int_{\Delta_\sigma} d\widehat{\mu}(\tau) = \delta^2.$$

We have

$$A\widehat{\tau}^{2r} + B\sigma^{2r} = 1, \quad A = \frac{\delta^2}{2\pi}.$$

Hence

$$B = \frac{1}{\sigma^{2r}} - \frac{\delta^2}{2\pi}\left(\frac{k}{r}\right)^{\frac{r}{r-k}}.$$

It can be easily verified that the condition $B > 0$ is equivalent to the condition $\sigma < \widehat{\sigma}$. Since $\mathcal{L}(\widehat{\mu}, \widehat{\lambda}_1, \widehat{\lambda}_2) = 0$, we solve problem (46) for all $\sigma > 0$.

It follows from Theorem 2 that the value of the problem (46) coincides with the value of the problem

$$\int_{\mathbb{R}} \tau^{2k}\, d\mu(\tau) \to \max, \quad \int_{\mathbb{R}} (\widehat{\lambda}_1 \tau^{2r} + 2\pi\widehat{\lambda}_2 \chi_\sigma(\tau))\, d\mu(\tau) \le \widehat{\lambda}_1 + \widehat{\lambda}_2\delta^2.$$

Since delta functions can be approximate by step functions, the value of (44) coincides with the value of the problem

$$\int_{\mathbb{R}} \tau^{2k} u(\tau)\, d\tau \to \max, \quad \int_{\mathbb{R}} (\widehat{\lambda}_1 \tau^{2r} + 2\pi\widehat{\lambda}_2 \chi_\sigma(\tau) u(\tau)\, d\tau \le \widehat{\lambda}_1 + \widehat{\lambda}_2\delta^2,$$
$$u \in L_1(\mathbb{R}), \quad u(\tau) \ge 0 \text{ almost everywhere on } \mathbb{R},$$

Now it follows from Theorem 1 that it remains to find the solution of the extremal problem

$$\widehat{\lambda}_1 \|x^{(r)}\|^2_{L_2(\mathbb{R})} + \widehat{\lambda}_2 \|Fx - y\|^2_{L_2(\Delta_\sigma)} \to \min, \quad x \in \mathcal{W}_2^r(\mathbb{R}).$$

Passing to Fourier transforms and using the Plancherel theorem we obtain the following problem

$$\int_{\Delta_\sigma} \left(\frac{\widehat{\lambda}_1}{2\pi} \tau^{2r} |Fx(\tau)|^2 + \widehat{\lambda}_2 |Fx(\tau) - y(\tau)|^2 \right) d\tau +$$
$$\frac{\widehat{\lambda}_1}{2\pi} \int_{\mathbb{R}\setminus\Delta_\sigma} \tau^{2r} |Fx(\tau)|^2\, d\tau \to \min, \quad x \in \mathcal{W}_2^r(\mathbb{R}).$$

It can be easily verified that the solution of this problem is the function x_0 such that

$$Fx_0(\tau) = \begin{cases} \left(1 + \dfrac{\tau^{2r}\widehat{\lambda}_1}{2\pi\widehat{\lambda}_2}\right)^{-1} y(\tau), & \tau \in \Delta_\sigma, \\ 0, & \tau \notin \Delta_\sigma. \end{cases}$$

Thus an optimal method of recovery has the form

$$\widehat{m}(y) = \frac{1}{2\pi} \int_{\Delta_\sigma} (i\tau)^k \left(1 + \frac{\tau^{2r}\widehat{\lambda}_1}{2\pi\widehat{\lambda}_2}\right)^{-1} y(\tau) e^{it\tau}\, d\tau.$$

For the optimal recovery error we have the following equality

$$E_2^\sigma(D^k, W_2^r(\mathbb{R}), \delta) = \sqrt{\widehat{\lambda}_1 + \widehat{\lambda}_2\delta^2}.$$

For $\sigma \geq \widehat{\sigma}$ we have

$$\widehat{\lambda}_1 = \frac{k}{r}\left(\frac{\delta^2}{2\pi}\right)^{1-k/r}, \quad \widehat{\lambda}_2 = \frac{1}{2\pi}\frac{r-k}{r}\left(\frac{2\pi}{\delta^2}\right)^{k/r}.$$

Consequently, in this case

$$E_2^\sigma(D^k, W_2^r(\mathbb{R}), \delta) = \left(\frac{\delta}{\sqrt{2\pi}}\right)^{1-k/r}$$

and the method

$$\widehat{m}(y) = \int_{-\sigma}^{\sigma}(i\tau)^k\left(1+\frac{\delta^2}{2\pi}\frac{k}{r-k}\tau^{2r}\right)^{-1} y(\tau)e^{it\tau}\,d\tau$$

is optimal.

Lecture 18

Let us show that the method

$$\widehat{m}(y) = \int_{-\widehat{\sigma}}^{\widehat{\sigma}} (i\tau)^k \left(1 + \frac{\delta^2}{2\pi}\frac{k}{r-k}\tau^{2r}\right)^{-1} y(\tau)e^{it\tau}\,d\tau$$

is also optimal.

First, we note that for all $\sigma > \widehat{\sigma}$

$$E_2^{\sigma}(D^k, W_2^r(\mathbb{R}), \delta) = E_2^{\widehat{\sigma}}(D^k, W_2^r(\mathbb{R}), \delta).$$

Since $L_2(\Delta_\sigma) \subset L_2(\Delta_{\widehat{\sigma}})$ and for all $y \in L_2(\Delta_\sigma)$ such that $\|Fx - y\|_{L_2(\Delta_\sigma)} \le \delta$ the same inequality in $L_2(\Delta_{\widehat{\sigma}})$-norm holds, we have

$$\sup_{\substack{x\in W_2^r(\mathbb{R}),\ y\in L_2(\Delta_\sigma)\\ \|Fx-y\|_{L_2(\Delta_\sigma)}\le\delta}} \|x^{(k)} - \widehat{m}(y)\|_{L_2(\mathbb{R})} \le \sup_{\substack{x\in W_2^r(\mathbb{R}),\ y\in L_2(\Delta_{\widehat{\sigma}})\\ \|Fx-y\|_{L_2(\Delta_{\widehat{\sigma}})}\le\delta}} \|x^{(k)} - \widehat{m}(y)\|_{L_2(\mathbb{R})} = E_2^{\widehat{\sigma}}(D^k, W_2^r(\mathbb{R}), \delta) = E_2^{\sigma}(D^k, W_2^r(\mathbb{R}), \delta).$$

It means that the method $\widehat{m}$ is optimal.

Now consider the case $k = 0$. Then for the extended dual problem we have

$$\mathcal{L}(\mu, \widehat{\lambda}_1, \widehat{\lambda}_2) = \int_{\Delta_\sigma} (-1 + \widehat{\lambda}_1\tau^{2r} + 2\pi\widehat{\lambda}_2)\,d\mu(\tau) + \int_{\mathbb{R}\setminus\Delta_\sigma} (-1 + \widehat{\lambda}_1\tau^{2r})\,d\mu(\tau).$$

Put

$$\widehat{\lambda}_2 = \frac{1}{\sigma^{2r}}, \quad \widehat{\lambda}_2 = \frac{1}{2\pi}.$$

Then for all μ

$$\mathcal{L}(\mu, \widehat{\lambda}_1, \widehat{\lambda}_2) = \widehat{\lambda}_1 \int_{\Delta_\sigma} \tau^{2r}\,d\mu(\tau) + \int_{\mathbb{R}\setminus\Delta_\sigma} \left(-1 + \left(\frac{\tau}{\sigma}\right)^{2r}\right) d\mu(\tau) \ge 0.$$

For

$$d\widehat{\mu}(t) = \frac{\delta^2}{2\pi}\delta(t) + \frac{1}{\sigma^{2r}}\delta(t-\sigma)$$

the conditions

$$\int_{\mathbb{R}} \tau^{2r}\,d\widehat{\mu}(\tau) = 1, \quad 2\pi\int_{\Delta_\sigma} d\widehat{\mu}(\tau) = \delta^2$$

are fulfilled and $\mathcal{L}(\widehat{\mu}, \widehat{\lambda}_1, \widehat{\lambda}_2) = 0$. Similar to the arguments used above we obtain that

$$E_2^\sigma(D^0, W_2^r(\mathbb{R}), \delta) = \sqrt{\frac{\delta^2}{2\pi} + \frac{1}{\sigma^{2r}}} \tag{48}$$

and the method

$$\widehat{m}(y) = \frac{1}{2\pi} \int_{-\sigma}^{\sigma} \left(1 + \left(\frac{\tau}{\sigma}\right)^{2r}\right)^{-1} y(\tau) e^{i\tau t} \, d\tau \tag{49}$$

is optimal.

Thus, we prove

Theorem 11. *Let $r \in \mathbb{N}$, $0 < k < r$, $0 < \sigma \le \infty$, $\delta > 0$, and*

$$\widehat{\sigma} = \left(\frac{r}{k}\right)^{\frac{1}{2(r-k)}} \left(\frac{2\pi}{\delta^2}\right)^{\frac{1}{2r}}.$$

Then

$$E_2^\sigma(D^k, W_2^r(\mathbb{R}), \delta) = \begin{cases} \sigma^k \sqrt{\dfrac{r-k}{2\pi r} \left(\dfrac{k}{r}\right)^{\frac{k}{r-k}} \delta^2 + \dfrac{1}{\sigma^{2r}}}, & \sigma < \widehat{\sigma}, \\ \left(\dfrac{\delta}{\sqrt{2\pi}}\right)^{1-k/r}, & \sigma \ge \widehat{\sigma} \end{cases}$$

and the method

$$\widehat{m}(y) = \frac{1}{2\pi} \int_{-\sigma_0}^{\sigma_0} (i\tau)^k \left(1 + \frac{r}{r-k} \left(\frac{r}{k}\right)^{\frac{k}{r-k}} \left(\frac{\tau}{\sigma_0}\right)^{2r}\right)^{-1} y(\tau) e^{i\tau t} \, d\tau,$$

where $\sigma_0 = \min(\sigma, \widehat{\sigma})$, is optimal.

If $k = 0$ and $0 < \sigma < \infty$, then the error of optimal recovery is given by (48) *and method* (49) *is optimal.*

It follows from Theorem11 that for a given δ, starting from $\widehat{\sigma}$, further extension of the interval on which the Fourier transform of a function from $W_2^r(\mathbb{R})$ is given with error δ in the $L_2(\Delta_\sigma)$-metric does not result in a decrease in the recovery error. In other words, if the relation

$$\delta^2 \sigma^{2r} \le 2\pi \left(\frac{r}{k}\right)^{\frac{r}{r-k}} \tag{50}$$

between the input data and the size of the interval on which the data is measured is violated, then the available information turns out to be redundant. The inequality (50) may be considered as an uncertainly principle.

Lecture 19

12. Generalization of the main theorems

Now we want to consider the case when the approximation of Fourier transforms is given in the uniform norm. To obtain the appropriate results we need a generalization of main Theorems 1 and 2.

Let X be a linear space, $Y_1, \dots, Y_n$ be linear spaces with semi-inner products $(\cdot,\cdot)_{Y_j}$, $j = 1, \dots, n$, and the corresponding semi-norms $\|\cdot\|_{Y_j}$ ($\|x\|_{Y_j} = \sqrt{(x,x)_{Y_j}}$), $Y_s = L_\infty(\Delta_s)$, $\Delta_s \subseteq \mathbb{R}$, $s = n+1, \dots, p$, $I_j \colon X \to Y_j$, $j = 1, \dots, p$, be linear operators, and Z be a normed linear space. Assume that

$$\omega \subset \{1, 2, \dots, n\}, \quad \Omega = \{1, 2, \dots, n\} \setminus \omega,$$
$$\psi \subset \{n+1, n+2, \dots, p\}, \quad \Psi = \{n+1, n+2, \dots, p\} \setminus \psi.$$

We consider the problem of optimal recovery of the operator $T \colon X \to Z$ on the set

$$W_{\omega\psi} = \{\, x \in X : \|I_j x\|_{Y_j} \le \delta_j,\ j \in \omega,\ |I_s x(t) - y_s(t)| \le \delta_s(t),\ t \in \Delta_\sigma,\ s \in \psi \,\}$$

(if $\omega = \psi = \emptyset$ we take $W = X$) from the information about values of operators I_j, $j \in \Omega \cup \Psi$ given with errors. Throughout what follows for functions from $L_\infty(\Delta_s)$ we will not note each time that inequalities hold almost everywhere on Δ_s. Let

$$\mathcal{Y} = \prod_{j \in \Omega \cup \Psi} Y_j.$$

We assume that for any $x \in W$ we know the vector $y = \{y_j\} \in \mathcal{Y}$ such that

$$\|I_j x - y_j\|_{Y_j} \le \delta_j,\ j \in \Omega, \quad |I_s x(t) - y_s(t)| \le \delta_s(t),\ t \in \Delta_\sigma,\ s \in \Psi.$$

Knowing the vector y we want to recover Tx.

Any operator $m \colon \mathcal{Y} \to Z$ is admitted as a recovery method. According to the general setting the value

$$e(T, W_{\omega\psi}, I, \delta, m) = \sup_{x \in W_{\omega\psi}} \sup_{\substack{y = \{y_j\} \in \mathcal{Y} \\ \|I_j x - y_j\|_{Y_j} \le \delta_j,\ j \in \Omega \\ |I_s x(t) - y_s(t)| \le \delta_s(t),\ t \in \Delta_\sigma,\ s \in \Psi}} \|Tx - m(y)\|_Z$$

is called the error of recovery of the method m (here $I = (I_1, \dots, I_p)$, $\delta = (\delta_1, \dots \delta_p)$). The quantity

$$E(T, W_{\omega\psi}, I, \delta) = \inf_{m \colon \mathcal{Y} \to Z} e(T, W_{\omega\psi}, I, \delta, m)$$

is called the error of optimal recovery. A method delivering the lower bound is called optimal.

The formulated problem of optimal recovery is closely connected with the following extremal problem (we shall call it the *duality* extremal problem)

$$(51)\quad \|Tx\|^2_Z \to \max,\quad \|I_jx\|^2_{Y_j} \le \delta_j^2,\ j=1,\dots,n,$$
$$|I_sx(t)|^2 \le \delta_s^2(t),\ t\in\Delta_\sigma,\ s=n+1,\dots,p,\quad x\in X.$$

Theorem 12. *Suppose that there exist measurable nonnegative functions $\widehat{\lambda}_s$ on Δ_s, $s=n+1,\dots,p$, and $\widehat{\lambda}_j \ge 0$, $j=1,\dots,n$, such that the value of the extremal problem*

$$(52)\quad \|Tx\|^2_Z \to \max,\quad \sum_{j=1}^n \widehat{\lambda}_j\|I_jx\|^2_{Y_j} + \sum_{s=n+1}^p \int_{\Delta_s} \widehat{\lambda}_s(t)|I_sx(t)|^2\,dt \le S,$$
$$x\in X,$$

where

$$S = \sum_{j=1}^n \widehat{\lambda}_j\delta_j^2 + \sum_{s=n+1}^p \int_{\Delta_s} \widehat{\lambda}_s(t)\delta_s^2(t)\,dt$$

is the same as in (51). *Moreover, assume that for all $y=(y_1,\dots,y_p)\in Y_1\times\ldots\times Y_p$ there exists $x_y = x(y_1,\dots,y_p)$ which is a solution of the extremal problem*

(53)
$$\sum_{j=1}^n \widehat{\lambda}_j\|I_jx-y_j\|^2_{Y_j} + \sum_{s=n+1}^p \int_{\Delta_s} \widehat{\lambda}_s(t)|I_sx(t)-y_s(t)|^2\,dt \to \min,\quad x\in X.$$

Then for all ω and ψ

$$E(T,W_{\omega\psi},I,\delta) = \sup_{\substack{x\in X\\ \|I_jx\|_{Y_j}\le\delta_j,\ j=1,\dots,n\\ |I_sx(t)|\le\delta_s(t),\ t\in\Delta_\sigma,\ s=n+1,\dots,p}} \|Tx\|_Z$$

and the method

$$(54)\quad \widehat{m}(y) = Tx(\widehat{y}),$$

where

$$(55)\quad \widehat{y} = \{\widehat{y}_j\}_{j=1}^p,\quad \widehat{y}_j = \begin{cases} y_j, & j\in\Omega\cup\Psi,\\ 0, & j\in\omega\cup\psi,\end{cases}$$

is optimal.

Proof. From Lemma 2 immediately follows the lower bound

$$E(T,W_{\omega\psi},I,\delta)\geq \sup_{\substack{x\in X\\ \|I_jx\|_{Y_j}\leq\delta_j,\ j=1,\dots,n\\ |I_sx(t)|\leq\delta_s(t),\ t\in\Delta_\sigma,\ s=n+1,\dots,p}} \|Tx\|_Z. \tag{56}$$

The upper bound. Consider the linear space $E=Y_1\times\ldots\times Y_p$ with the semi-inner product

$$(y^1,y^2)_E=\sum_{j=1}^n\widehat{\lambda}_j(y^1_j,y^2_j)_{Y_j}+\sum_{s=n+1}^p\int_{\Delta_s}\widehat{\lambda}_s(t)y^1_s(t)\overline{y^2_s(t)}\,dt,$$

where $y^1=(y^1_1,\dots,y^1_p)$, $y^2=(y^2_1,\dots,y^2_p)$. Now the extremal problem (53) can be rewritten in the form

$$\|\widetilde{I}x-y\|^2_E\to\max,\quad x\in X,$$

where $\widetilde{I}x=(I_1x,\dots,I_px)$ and $y=(y_1,\dots,y_p)$. It follows from Proposition 1 that for all $x\in X$

$$(\widetilde{I}x_y-y,\widetilde{I}x)_E=0.$$

Consequently,

$$\|\widetilde{I}x-y\|^2_E=\|\widetilde{I}x-\widetilde{I}x_y\|^2_E+\|\widetilde{I}x_y-y\|^2_E.$$

Indeed, we have

$$\begin{aligned}\|\widetilde{I}x-y\|^2_E&=\|\widetilde{I}x-\widetilde{I}x_y+\widetilde{I}x_y-y\|^2_E\\&=\|\widetilde{I}x-\widetilde{I}x_y\|^2_E-2\operatorname{Re}(\widetilde{I}x-\widetilde{I}x_y,\widetilde{I}x_y-y)_E+\|\widetilde{I}x_y-y\|^2_E\\&=\|\widetilde{I}x-\widetilde{I}x_y\|^2_E+\|\widetilde{I}x_y-y\|^2_E.\end{aligned}$$

Thus, for all $x\in X$

$$\begin{aligned}\|\widetilde{I}x-\widetilde{I}x_y\|^2_E\leq\|\widetilde{I}x-y\|^2_E&=\sum_{j=1}^n\widehat{\lambda}_j\|I_jx-y_j\|^2_{Y_j}\\&+\sum_{s=n+1}^p\int_{\Delta_s}\widehat{\lambda}_s(t)|I_sx(t)-y_s(t)|^2\,dt.\end{aligned}\tag{57}$$

Let $x\in W_{\omega\psi}$, $y=\{y_j\}\in\mathcal{Y}$ such that

$$\|I_jx-y_j\|_{Y_j}\leq\delta_j,\ j\in\Omega,\quad |I_sx(t)-y_s(t)|\leq\delta_s(t),\ t\in\Delta_\sigma,\ s\in\Psi,$$

and $\widehat{y}$ be defined by (55). Put $z=x-x_{\widehat{y}}$. Then it follows from (57) that

$$\sum_{j=1}^n\widehat{\lambda}_j\|I_jz\|^2_{Y_j}+\sum_{s=n+1}^p\int_{\Delta_s}\widehat{\lambda}_s(t)|I_sz(t)|^2\,dt=\|\widetilde{I}z\|^2_E\leq S.$$

Now for the method (54) we have the following estimate

$$\|Tx - \widehat{m}(y)\|_Z^2 = \|Tz\|_Z^2$$

$$\leq \sup\left\{ \|Tz\|_Z^2 : \sum_{j=1}^{n} \widehat{\lambda}_j \|I_j z\|_{Y_j}^2 + \sum_{s=n+1}^{p} \int_{\Delta_s} \widehat{\lambda}_s(t) |I_s z(t)|^2 \, dt \leq S \right\}$$

$$= \sup_{\substack{x \in X \\ \|I_j x\|_{Y_j} \leq \delta_j, \ j=1,\dots,n \\ |I_s x(t)| \leq \delta_s(t), \ t \in \Delta_\sigma, \ s=n+1,\dots,p}} \|Tx\|_Z^2.$$

Consequently,

$$E(T, W_{\omega\psi}, I, \delta) \leq \sup_{\substack{x \in X \\ \|I_j x\|_{Y_j} \leq \delta_j, \ j=1,\dots,n \\ |I_s x(t)| \leq \delta_s(t), \ t \in \Delta_\sigma, \ s=n+1,\dots,p}} \|Tx\|_Z.$$

Taking into account the lower bound (56), we obtain that

$$E(T, W_{\omega\psi}, I, \delta) = \sup_{\substack{x \in X \\ \|I_j x\|_{Y_j} \leq \delta_j, \ j=1,\dots,n \\ |I_s x(t)| \leq \delta_s(t), \ t \in \Delta_\sigma, \ s=n+1,\dots,p}} \|Tx\|_Z$$

and $\widehat{m}$ is an optimal method. □

Lecture 20

Now we obtain a sufficient conditions for coinciding the values of problems (51) and (52) which are similar to the ones that were obtained in Theorem 2. Put

$$\mathcal{L}(x,\lambda) = -\|Tx\|_Z^2 + \sum_{j=1}^{n} \lambda_j \|I_j x\|_{Y_j}^2 + \sum_{s=n+1}^{p} \int_{\Delta_s} \lambda_s(t)|I_s x(t)|^2\,dt$$

(here $\lambda = (\lambda_1, \dots, \lambda_p)$. $\mathcal{L}$ is the so-called the Lagrange function for the extremal problem (51). We call $\widehat{x} \in X$ an extremal element if it is admissible in (51) (that is, $\|I_j x\|_{Y_j}^2 \le \delta_j^2$, $j = 1, \dots, n$, $|I_s x(t)|^2 \le \delta_s^2(t)$, $t \in \Delta_\sigma$, $s = n+1, \dots, p$) and

$$\|T\widehat{x}\|_Z^2 = \sup_{\substack{x \in X \\ \|I_j x\|_{Y_j} \le \delta_j,\ j=1,\dots,n \\ |I_s x(t)| \le \delta_s(t),\ t \in \Delta_\sigma,\ s=n+1,\dots,p}} \|Tx\|_Z^2.$$

Theorem 13 (sufficient condition)**.** *Suppose that there exist measurable nonnegative functions $\widehat{\lambda}_s$ on Δ_s, $s = n+1, \dots, p$, nonnegative real numbers $\widehat{\lambda}_j$, $j = 1, \dots, n$, and $\widehat{x} \in X$ admissible in (51) such that*

$$(a) \quad \min_{x \in X} \mathcal{L}(x, \widehat{\lambda}) = \mathcal{L}(\widehat{x}, \widehat{\lambda}), \quad \widehat{\lambda} = (\widehat{\lambda}_1, \dots, \widehat{\lambda}_p),$$

$$(b) \quad \sum_{j=1}^{n} \widehat{\lambda}_j (\|I_j \widehat{x}\|_{Y_j}^2 - \delta_j^2) + \sum_{s=n+1}^{p} \int_{\Delta_s} \widehat{\lambda}_s(t)(|I_s \widehat{x}(t)|^2 - \delta_s^2(t))\,dt = 0.$$

Then $\widehat{x}$ is an extremal element and

$$\sup_{\substack{x \in X \\ \|I_j x\|_{Y_j} \le \delta_j,\ j=1,\dots,n \\ |I_s x(t)| \le \delta_s(t),\ t \in \Delta_\sigma,\ s=n+1,\dots,p}} \|Tx\|_Z^2 = \sup\left\{ \|Tz\|_Z^2 : \sum_{j=1}^{n} \widehat{\lambda}_j \|I_j z\|_{Y_j}^2 + \sum_{s=n+1}^{p} \int_{\Delta_s} \widehat{\lambda}_s(t)|I_s z(t)|^2\,dt \le S \right\} = S.$$

Proof. Let $x \in X$ be an admissible element in (51). Then

$$-\|Tx\|_Z^2 \ge -\|Tx\|_Z^2 + \sum_{j=1}^{n} \widehat{\lambda}_j(\|I_jx\|_{Y_j}^2 - \delta_j^2)$$
$$+ \sum_{s=n+1}^{p} \int_{\Delta_s} \widehat{\lambda}_s(t)(|I_sx(t)|^2 - \delta_s^2(t))\,dt = \mathcal{L}(x, \widehat{\lambda}) - S$$
$$\ge \mathcal{L}(\widehat{x}, \widehat{\lambda}) - S = -\|T\widehat{x}\|_Z^2 + \sum_{j=1}^{n} \widehat{\lambda}_j(\|I_j\widehat{x}\|_{Y_j}^2 - \delta_j^2)$$
$$+ \sum_{s=n+1}^{p} \int_{\Delta_s} \widehat{\lambda}_s(t)(|I_s\widehat{x}(t)|^2 - \delta_s^2(t))\,dt) = -\|T\widehat{x}\|_Z^2.$$

The same arguments show that $\widehat{x}$ is an extremal element in the problem (52). The proof of the equality $\mathcal{L}(\widehat{x}, \widehat{\lambda}) = 0$ is the same as in Theorem 2. Now we have

$$\sup_{\substack{x \in X \\ \|I_jx\|_{Y_j} \le \delta_j,\ j=1,\dots,n \\ |I_sx(t)| \le \delta_s(t),\ t \in \Delta_\sigma,\ s=n+1,\dots,p}} \|Tx\|_Z^2 = \|T\widehat{x}\|_Z^2 = -\mathcal{L}(\widehat{x}, \widehat{\lambda}) + S = S.$$

□

13. Optimal recovery of derivatives from Fourier transforms given with an error in the uniform norm

Recall that the space $\mathcal{F}_{2,\infty}^r$ is the set of all functions x such that $x^{(r-1)}$ is locally absolute continues on $\mathbb{R}$, $x^{(r)} \in L_2(\mathbb{R})$, and $Fx \in L_\infty(\mathbb{R})$. $F_{2,\infty}^r$ is the set of functions $x \in \mathcal{F}_{2,\infty}^r$ for which $\|x^{(r)}\|_{L_2(\mathbb{R})} \le 1$. Now we consider the problem of optimal recovery of $x^{(k)}$, $0 \le k < r$, on the class $F_{2,\infty}^r$ from the Fourier transform of x given approximately on a finite interval $\Delta_\sigma = (-\sigma, \sigma)$, $0 < \sigma \le \infty$, when the error is measured in the uniform norm.

Assume that for any $x \in F_{2,\infty}^r$ we know $y \in L_\infty(\Delta_\sigma)$ such that

$$|Fx(t) - y(t)| \le \delta(t), \quad t \in \Delta_\sigma.$$

Knowing y we have to recover $x^{(k)}$. We define the error of optimal recovery by

$$E_\infty^\sigma(D^k, F_{2,\infty}^r, \delta) = \inf_{m\colon L_\infty(\Delta_\sigma) \to L_2(\mathbb{R})} \sup_{\substack{x \in F_{2,\infty}^r,\ y \in L_\infty(\Delta_\sigma) \\ |Fx(t) - y(t)| \le \delta(t),\ t \in \Delta_\sigma}} \|x^{(k)} - m(y)\|_{L_2(\mathbb{R})}.$$

Theorem 14. *Let* $r \in \mathbb{N}$, $k \in \mathbb{Z}_+$, $0 \le k < r$, $0 < \sigma \le \infty$, $\delta \in L_\infty(\Delta_\sigma)$, $\delta(t) \ge 0$, *and*

$$\sigma_0 = \sup \left\{ a : 0 < a < \sigma,\ \frac{1}{2\pi} \int_{-a}^{a} t^{2r} \delta^2(t)\, dt \le 1 \right\}.$$

If $\sigma_0 < \infty$, *then*

$$E_\infty^\sigma(D^k, F_{2,\infty}^r, \delta) = \sqrt{\sigma_0^{-2(r-k)} + \frac{1}{2\pi} \int_{-\sigma_0}^{\sigma_0} (t^{2k} - \sigma_0^{-2(r-k)} t^{2r}) \delta^2(t)\, dt}$$

and the method

$$\widehat{m}(y) = \frac{1}{2\pi} \int_{-\sigma_0}^{\sigma_0} (i\tau)^k \left(1 - \left(\frac{\tau}{\sigma_0} \right)^{2(r-k)} \right) y(\tau) e^{i\tau t}\, d\tau \tag{58}$$

is optimal.

If $\sigma_0 = \infty$, *then*

$$E_\infty^\sigma(D^k, F_{2,\infty}^r, \delta) = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} t^{2k} \delta^2(t)\, dt}$$

and the method

$$\widehat{m}(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\tau)^k y(\tau) e^{i\tau t}\, d\tau \tag{59}$$

is optimal.

Proof. In this case the dual problem has the form

$$\|x^{(k)}\|_{L_2(\mathbb{T})}^2 \to \max, \quad \|x^{(r)}\|_{L_2(\mathbb{R})}^2 \le 1, \quad |Fx(t)|^2 \le \delta^2(t),\ t \in \Delta_\sigma,$$
$$x \in \mathcal{F}_{2,\infty}^r.$$

The Lagrange function has the form

$$\mathcal{L}(x, \lambda_1, \lambda_2) = -\|x^{(k)}\|_{L_2(\mathbb{R})}^2 + \lambda_1 \|x^{(r)}\|_{L_2(\mathbb{R})}^2 + \int_{\Delta_\sigma} \lambda_2(t) |Fx(t)|^2\, dt.$$

Passing to Fourier transforms and writing $(2\pi)^{-1}|Fx|^2 = u$, we have

$$\mathcal{L}(x, \lambda_1, \lambda_2) = \int_{\Delta_\sigma} \left(-t^{2k} + \lambda_1 t^{2r} + 2\pi \lambda_2(t) \right) u(t)\, dt + \int_{\mathbb{R} \setminus \Delta_\sigma} \left(-t^{2k} + \lambda_1 t^{2r} \right) u(t)\, dt$$

by the Plancherel theorem.

First, let $\sigma_0 < \infty$. Let $\widehat{\lambda}_1 = \sigma_0^{-2(r-k)}$ and

$$\widehat{\lambda}_2(t) = \begin{cases} (2\pi)^{-1}\left(t^{2k} - \widehat{\lambda}_1 t^{2r}\right), & |t| < \sigma_0, \\ 0, & |t| \geq \sigma_0. \end{cases}$$

Then

$$\mathcal{L}(x, \widehat{\lambda}_1, \widehat{\lambda}_2) = \int_{|t|\geq\sigma_0} \left(-t^{2k} + \sigma_0 t^{2r}\right) u(t)\, dt \geq 0$$

for all $x \in \mathcal{F}^r_{2,\infty}$.

Set

$$\gamma = 1 - \frac{1}{2\pi}\int_{-\sigma_0}^{\sigma_0} t^{2r}\delta^2(t)\, dt.$$

If $\gamma = 0$, we define $\widehat{x}$ from the condition

$$F\widehat{x}(t) = \begin{cases} \delta(t), & |t| < \sigma_0, \\ 0, & |t| \geq \sigma_0. \end{cases}$$

Then $\mathcal{L}(\widehat{x}, \widehat{\lambda}_1, \widehat{\lambda}_2) = 0$,

$$\|\widehat{x}^{(r)}\|^2_{L_2(\mathbb{R})} = \frac{1}{2\pi}\int_{-\sigma_0}^{\sigma_0} t^{2r}\delta^2(t)\, dt = 1.$$

Moreover, it is easy to see that

$$\int_{\Delta_\sigma} \widehat{\lambda}_2(t)(|F\widehat{x}(t)|^2 - \delta^2(t))\, dt = 0.$$

It means that conditions (a) and (b) of Theorem 13 are fulfilled.

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If $\gamma > 0$ (in this case, it is obvious that $\sigma_0 = \sigma$), then we set

$$F\widehat{x}(t) = \begin{cases} \delta(t), & |t| < \sigma, \\ \sqrt{A\Delta(t-\sigma)}, & |t| \geq \sigma, \end{cases}$$

where $\Delta(t - t_0)$ is the delta function with the unit mass concentrated at t_0, $A > 0$. In this case $\mathcal{L}(\widehat{x}, \widehat{\lambda}_1, \widehat{\lambda}_2) = 0$ and

$$\|\widehat{x}^{(r)}\|^2_{L_2(\mathbb{R})} = \frac{1}{2\pi}\int_{-\sigma}^{\sigma} t^{2r}\delta^2(t)\,dt + \frac{1}{2\pi}A\sigma^{2r}.$$

Taking

$$A = 2\pi\sigma^{-2r}\left(1 - \frac{1}{2\pi}\int_{-\sigma}^{\sigma} t^{2r}\delta^2(t)\,dt\right),$$

we obtain that conditions (a) and (b) of Theorem 13 are fulfilled.

To obtain an optimal method of recovery we have to solve the following extremal problem

$$\widehat{\lambda}_1\|x^{(r)}\|^2_{L_2(\mathbb{R})} + \int_{\Delta_\sigma} \widehat{\lambda}_2(t)|Fx(t) - y(t)|^2\,dt \to \max, \quad x \in \mathcal{F}^r_{2,\infty}.$$

Passing to the Fourier transform we get

$$\int_{\Delta_\sigma}\left(\frac{\widehat{\lambda}_1}{2\pi}t^{2r}|Fx(t)|^2 + \widehat{\lambda}_2(t)|Fx(t) - y(t)|^2\right)\,dt \to \max, \quad x \in \mathcal{F}^r_{2,\infty}.$$

It is easy to obtain the solution of this problem

$$Fx_y(t) = \begin{cases} \dfrac{2\pi\widehat{\lambda}_2(t)}{\widehat{\lambda}_1 t^{2r} + 2\pi\widehat{\lambda}_2(t)}y(t), & |t| < \sigma_0, \\ 0, & |t| \geq \sigma_0. \end{cases}$$

That is,

$$Fx_y(t) = \begin{cases} \left(1 - \left(\dfrac{t}{\sigma_0}\right)^{2(r-k)}\right)y(t), & |t| < \sigma_0, \\ 0, & |t| \geq \sigma_0. \end{cases}$$

Now for the considered case the result of the theorem immediately follows from Theorem 12.

If $\sigma_0 = \infty$ (in this case, obviously, $\sigma = \infty$), then it follows from Lemma 2 that

$$E_\infty^\sigma(D^k, F_{2,\infty}^r, \delta) \geq \sup_{\substack{x\in F_{2,\infty}^r \\ |Fx(t)|\leq\delta(t),\ t\in\mathbb{R}}} \|x^{(k)}\|_{L_2(\mathbb{R})}$$

$$\geq \|\widehat{x}^{(k)}\|_{L_2(\mathbb{R})} = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} t^{2k}\delta^2(t)\,dt},$$

where $\widehat{x}$ is the inverse Fourier transform of δ. On the other hand,

$$e_\infty^\sigma(D^k, F_{2,\infty}^r, \delta, \widehat{m}) = \sup_{\substack{x\in F_{2,\infty}^r,\ y\in L_2(\mathbb{R}) \\ |Fx(t)-y(t)|\leq\delta(t),\ t\in\mathbb{R}}} \|x^{(k)} - \widehat{m}(y)\|_{L_2(\mathbb{R})}$$

$$= \sup_{\substack{x\in F_{2,\infty}^r,\ y\in L_2(\mathbb{R}) \\ |Fx(t)-y(t)|\leq\delta(t),\ t\in\mathbb{R}}} \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} t^{2k}|Fx(t)-y(t)|^2\,dt\right)^{1/2}$$

$$\leq \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} t^{2k}\delta^2(t)\,dt}$$

for the method (59). □

Corollary 3. *Let* $\delta(t) \equiv \delta > 0$ *and*

$$\widehat{\sigma} = (\pi(2r+1))^{\frac{1}{2r+1}}\delta^{-\frac{2}{2r+1}}.$$

Then

$$E_\infty^\sigma(D^k, F_{2,\infty}^r, \delta) = \begin{cases} \sqrt{\sigma^{-2(r-k)} + \dfrac{2\delta^2(r-k)}{\pi(2k+1)(2r+1)}\sigma^{2k+1}}, & \sigma < \widehat{\sigma}, \\ \sqrt{\dfrac{2r+1}{2k+1}}\left(\dfrac{1}{\pi(2r+1)}\right)^{\frac{r-k}{2r+1}}\delta^{\frac{2(r-k)}{2r+1}}, & \sigma \geq \widehat{\sigma}, \end{cases}$$

and the method (58) *with* $\sigma_0 = \min(\sigma, \widehat{\sigma})$ *is optimal.*

It follows from this corollary that for a given δ, starting from $\widehat{\sigma}$, further extension of the interval on which the Fourier transform of a function from in $F_{2,\infty}^r$ is given with error δ in the uniform metric does not result in a decrease in the recovery error. In other words, if the relation

$$\delta^2\sigma^{2r+1} \leq \pi(2r+1)$$

between the input data and the size of the interval on which the data is measured is violated, then the available information turns out to be redundant.

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From Corollary 2 and Corollary 3 we obtain

Corollary 4.

$$K_F(k,r,2,\infty,2)=\sqrt{\frac{2r+1}{2k+1}}\left(\frac{1}{\pi(2r+1)}\right)^{\frac{r-k}{2r+1}}$$

Thus, we obtained the exact inequality

$$\|x^{(k)}\|_{L_2(\mathbb{R})}\le\sqrt{\frac{2r+1}{2k+1}}\left(\frac{1}{\pi(2r+1)}\right)^{\frac{r-k}{2r+1}}\|Fx\|_{L_\infty(\mathbb{R})}^{\frac{2(r-k)}{2r+1}}\|x^{(r)}\|_{L_2(\mathbb{R})}^{\frac{2k+1}{2r+1}}.$$

14. Optimal recovery of derivatives in $\mathbb{R}^d$

First we recall some facts about the Fourier transform in $\mathbb{R}^d$. Let $x\in L_2(\mathbb{R}^d)$. Then the Fourier transform of the function x is defined as follows

$$Fx(\tau)=\int_{\mathbb{R}^d}x(t)e^{-i\langle\tau,t\rangle}\,dt,$$

where $\tau=(\tau_1,\ldots,\tau_d)$, $t=(t_1,\ldots,t_d)$, $\langle\tau,t\rangle=\tau_1t_1+\ldots+\tau_dt_d$. It follows from the Plancherel theorem that Fx can be considered as a function from $L_2(\mathbb{R}^d)$, moreover,

$$\|x\|^2_{L_2(\mathbb{R}^d)}=\frac{1}{(2\pi)^d}\|Fx\|^2_{L_2(\mathbb{R}^d)}.$$

The inverse Fourier transform is given by the formula

$$x(t)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}Fx(\tau)e^{i\langle t,\tau\rangle}\,d\tau.$$

For $x\in L_2(\mathbb{R}^d)$ we denote by $D^\alpha x$ the Weyl derivative of order α which is defined by

$$D^\alpha x(t)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}(i\tau)^\alpha Fx(\tau)e^{i\langle\tau,t\rangle}\,d\tau,$$

where

$$(i\tau)^\alpha=(i\tau_1)^{\alpha_1}\ldots(i\tau_d)^{\alpha_d}.$$

The Sobolev space $\mathcal{H}_2^r(\mathbb{R}^d)$, $r\ge1$, is the set of functions $x\in L_2(\mathbb{R}^d)$ such that

$$\|x\|_{\mathcal{H}_2^r(\mathbb{R}^d)}=\left(\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\left(1+\|t\|^2\right)^r|Fx(t)|^2\,dt\right)^{1/2}<\infty,$$

where $\|t\|^2=t_1^2+\ldots+t_d^2$. The Sobolev class is the set of functons

$$H_2^r(\mathbb{R}^d)=\{\,x\in\mathcal{H}_2^r(\mathbb{R}^d):\|x\|_{\mathcal{H}_2^r(\mathbb{R}^d)}\le1\,\}.$$

We state the problem on optimal recovery of $D^\alpha x$ on the class $H_2^r(\mathbb{R}^d)$ in the $L_2(\mathbb{R}^d)$-metric from the information about approximate values of Fourier transform Fx. Assume that for any $x \in H_2^r(\mathbb{R}^d)$ we know a function $y \in L_2(\mathbb{R}^d)$ such that

$$\|Fx - y\|_{L_2(\mathbb{R}^d)} \le \delta.$$

Knowing y we want to recover D^α.

We define the error of optimal recovery as follows

$$E_2(D^\alpha, H_2^r(\mathbb{R}^d), \delta) = \inf_{m\colon L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \sup_{\substack{x\in H_2^r(\mathbb{R}^d),\ y\in L_2(\mathbb{R}^d)\\ \|Fx-y\|_{L_2(\mathbb{R}^d)}\le\delta}} \|D^\alpha x - m(y)\|_{L_2(\mathbb{R}^d)}.$$

Any method for which the infimum is attained we call an optimal method of recovery.

Consider the duality problem

$$\|D^\alpha x\|^2_{L_2(\mathbb{R}^d)} \to \max, \quad \|Fx\|^2_{L_2(\mathbb{R}^d)} \le \delta^2, \quad \|x\|^2_{\mathcal{H}_2^r(\mathbb{R}^d)} \le 1.$$

Passing to Fourier transforms and using the Plancherel theorem, we may rewrite this problem in the form

$$(60) \quad \int_{\mathbb{R}^d} |t|^{2\alpha} u(t)\, dt \to \max, \quad (2\pi)^d \int_{\mathbb{R}^d} u(t)\, dt \le \delta^2, \quad \int_{\mathbb{R}^d} \left(1 + \|t\|^2\right)^r u(t)\, dt \le 1, \quad u(t) \ge 0,$$

where $|t|^{2\alpha} = |t_1|^{2\alpha_1} \dots |t_d|^{2\alpha_d}$ and

$$u = (2\pi)^{-d} |Fx|^2.$$

There is no existence of extremal function in this problem. Therefore, we consider the extension of this problem for measures

$$(61) \quad \int_{\mathbb{R}^d} |t|^{2\alpha}\, d\mu(t) \to \max, \quad (2\pi)^d \int_{\mathbb{R}^d} d\mu(t) \le \delta^2, \quad \int_{\mathbb{R}^d} \left(1 + \|t\|^2\right)^r\, d\mu(t) \le 1.$$

The Lagrange function for this problem has the form

$$\mathcal{L}(\mu, \lambda_1, \lambda_2) = \int_{\mathbb{R}^d} \left(-|t|^{2\alpha} + (2\pi)^d \lambda_1 + \lambda_2 \left(1 + \|t\|^2\right)^r\right)\, d\mu(t).$$

Consider the function

$$G(t) = -|t|^{2\alpha} + (2\pi)^d \lambda_1 + \lambda_2 \left(1 + \|t\|^2\right)^r.$$

First, we suppose that $\alpha_j > 0$. For $|t| > 0$ we put $\xi_j = 2\ln|t_j|$, $j = 1, \dots, d$. Then

$$G(t) = e^{\langle \alpha, \xi \rangle} F(\xi),$$

where $\xi = (\xi_1, \dots, \xi_d)$ and

$$F(\xi) = -1 + e^{-\langle \alpha, \xi \rangle} \left((2\pi)^d \lambda_1 + \lambda_2 \left(1 + e^{\xi_1} + \ldots + e^{\xi_d} \right)^r \right).$$

We show that F is a convex function for all $\lambda_1, \lambda_2 \geq 0$. The function F may be represented as follows

$$F(\xi) = -1 + (2\pi)^d \lambda_1 f(\xi) + \lambda_2 g^r(\xi),$$

where

$$f(\xi) = e^{-\langle \alpha, \xi \rangle}, \quad g(\xi) = \sum_{j=0}^{d} e^{\langle b_j, \xi \rangle}, \quad b_j = -\frac{1}{r}\alpha + e_j,$$

$$j = 0, \dots, d, \quad e_0 = (0, \dots, 0), \quad (e_j)_k = \begin{cases} 1, & k = j, \\ 0, & k \neq j, \end{cases}, \ j = 1, \dots, d.$$

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We have

$$d^2 g^r(\xi) = r(r-1)g^{r-2}(\xi)\bigg(\sum_{j=0}^{d} e^{\langle b_j,\xi\rangle}\langle b_j,\xi\rangle\bigg)^2$$

$$+\, r g^{r-1}(\xi)\sum_{j=0}^{d} e^{\langle b_j,\xi\rangle}\langle b_j,\xi\rangle^2 \geq 0, \quad d^2 f(\xi) = e^{-\langle\alpha,\xi\rangle}\langle\alpha,\xi\rangle^2 \geq 0.$$

Consequently, $d^2F(\xi) \geq 0$. It means that F is convex.

Define $\widehat{\xi} = (\widehat{\xi}_1,\dots,\widehat{\xi}_d)$ from the condition

$$e^{\widehat{\xi}_j} = c\alpha_j, \quad j = 1,\dots,d,$$

where $c > 0$ will be defined later, and find $\widehat{\lambda}_1$, $\widehat{\lambda}_2$ such that

$$F(\widehat{\xi}) = 0, \quad dF(\widehat{\xi}) = 0. \tag{62}$$

Set

$$\sigma = \sum_{j=1}^{d}\alpha_j, \quad p = \prod_{j=1}^{d}\alpha_j^{\alpha_j}.$$

Then

$$e^{\langle\alpha,\xi\rangle} = \prod_{j=1}^{d}(e^{\widehat{\xi}})^{\alpha_j} = pc^{\sigma}.$$

Consequently,

$$F(\widehat{\xi}) = -1 + \frac{1}{p}c^{-\sigma}\left((2\pi)^d\lambda_1 + \lambda_2(1+c\sigma)^r\right).$$

We have

$$\frac{\partial F}{\partial\xi_j}\Big|_{\xi=\widehat{\xi}} = -e^{-\langle\alpha,\widehat{\xi}\rangle}\alpha_j\left((2\pi)^d\lambda_1 + \lambda_2(1+c\sigma)^r - cr\lambda_2(1+c\sigma)^{r-1}\right).$$

To satisfy (62) we obtain the following equalities

$$(2\pi)^d\widehat{\lambda}_1 + \widehat{\lambda}_2(1+c\sigma)^r = pc^{\sigma},$$
$$(2\pi)^d\widehat{\lambda}_1 + \widehat{\lambda}_2(1+c\sigma)^r = cr\widehat{\lambda}_2(1+c\sigma)^{r-1}.$$

Assume that $\sigma < r$ and

$$c > \frac{1}{r-\sigma}. \tag{63}$$

Then

$$(64)\qquad \begin{aligned} \widehat{\lambda}_1 &= \frac{pc^{\sigma-1}}{(2\pi)^d r}(c(r-\sigma)-1) > 0, \\ \widehat{\lambda}_2 &= \frac{pc^{\sigma-1}}{r(1+c\sigma)^{r-1}} > 0. \end{aligned}$$

Conditions (62) together with convexity of F yield that $F(\xi) \geq 0$ for all $\xi \in \mathbb{R}^d$. Consequently, $G(t) \geq 0$ for all $t \in \mathbb{R}^d$ and $G(\widehat{\tau}) = 0$, where $\widehat{\tau} = (\widehat{\tau}_1, \ldots, \widehat{\tau}_d)$,

$$\widehat{\tau}_j = \sqrt{c\alpha_j}, \quad j = 1, \ldots, d.$$

If $\alpha_j > 0$, $j \in \Omega \subset \{1, \ldots, d\}$ and $\alpha_j = 0$, $j \in \Omega \setminus \{1, \ldots, d\}$, then the similar arguments show that for the function

$$\widetilde{G}(t) = -|t|^{2\alpha} + (2\pi)^d \lambda_1 + \lambda_2 \left(1 + \sum_{j\in\Omega} t_j^2\right)^r$$

$\widetilde{G}(\widehat{\tau}) = 0$ and $\widetilde{G}(t) \geq 0$ for all $t \in \mathbb{R}^d$. But in this case $G(t) \geq \widetilde{G}(t) \geq 0$ for all $t \in \mathbb{R}^d$ and $G(\widehat{\tau}) = \widetilde{G}(\widehat{\tau}) = 0$.

Put $d\widehat{\mu}(t) = A\delta(t - \widehat{\tau})$, where $\delta(t)$ is the delta function at the origin. Then

$$\min_{d\mu \geq 0} \mathcal{L}(d\mu, \widehat{\lambda}_1, \widehat{\lambda}_2) = \mathcal{L}(d\widehat{\mu}, \widehat{\lambda}_1, \widehat{\lambda}_2).$$

Define A from the conditions

$$(2\pi)^d \int_{\mathbb{R}^d} d\widehat{\mu}(t) = \delta^2, \quad \int_{\mathbb{R}^d} \left(1 + \|t\|^2\right)^r d\widehat{\mu}(t) = 1.$$

We have

$$(2\pi)^d A = \delta^2, \quad A\left(1 + \|\widehat{\tau}\|^2\right)^r = 1.$$

Hence

$$A = \Delta^2, \quad c = \frac{1}{\sigma}\left(\Delta^{-2/r} - 1\right),$$

where

$$\Delta = \frac{\delta}{(2\pi)^{d/2}}.$$

From (63) we obtain that

$$\delta < (2\pi)^{d/2}\Delta_0, \quad \Delta_0 = \left(1 - \frac{\sigma}{r}\right)^{r/2}.$$

If $\delta \geq (2\pi)^{d/2}\Delta_0$, we put

$$c = \frac{1}{r-\sigma} \quad A = \frac{1}{\left(1 + \|\tau\|^2\right)^r} = \Delta_0^2.$$

Then

$$(2\pi)^d \int_{\mathbb{R}^d} d\widehat{\mu}(t) = (2\pi)^d \Delta_0^2 \le \delta^2,$$

which means that $d\widehat{\mu}(t)$ is an admissible measure. Note that in this case $\widehat{\lambda}_1 = 0$.

To find an optimal method of recovery consider the extremal problem

$$\widehat{\lambda}_1 \|Fx - y\|^2_{L_2(\mathbb{R}^d)} + \widehat{\lambda}_2 \|x\|^2_{\mathcal{H}_2^r(\mathbb{R}^d)} \to \min, \quad x \in \mathcal{H}_2^r(\mathbb{R}^d).$$

Passing to the Fourier transform we have

$$\int_{\mathbb{R}^d} \left(\widehat{\lambda}_1 |Fx(t) - y(t)|^2 + \frac{\widehat{\lambda}_2}{(2\pi)^d} (1 + \|t\|^2)^r |Fx(t)|^2 \right) dt \to \min, \qquad x \in \mathcal{H}_2^r(\mathbb{R}^d).$$

It can be easily obtained that the solution of this problem has the form

$$Fx_y(t) = \frac{(2\pi)^d \widehat{\lambda}_1}{(2\pi)^d \widehat{\lambda}_1 + \widehat{\lambda}_2 (1 + \|t\|^2)^r} y(t).$$

If $\delta < (2\pi)^{d/2} \Delta_0$, then

$$\frac{\widehat{\lambda}_2}{(2\pi)^d \widehat{\lambda}_1} = \frac{1}{(1 + c\sigma)^{r-1}(c(r - \sigma) - 1)} = \frac{\Delta^{2-2/r}}{\frac{r - \sigma}{\sigma} (\Delta^{2/r} - 1) - 1}$$
$$= \frac{\Delta^2}{\frac{r - \sigma}{\sigma} - \frac{r}{\sigma} \Delta^{2/r}} = \frac{\sigma \Delta^2}{r(\Delta_0^{2/r} - \Delta^{2/r})}.$$

Thus for $\delta < (2\pi)^{d/2} \Delta_0$ the method

$$\widehat{m}(y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{(i\tau)^\alpha y(\tau) e^{i\langle \tau, t \rangle}}{1 + \frac{\sigma \Delta^2}{r(\Delta_0^{2/r} - \Delta^{2/r})} (1 + \|\tau\|^2)^r} \, d\tau \tag{65}$$

is optimal and the error of optimal recovery can be calculated as follows

$$
\begin{aligned}
E_2(D^\alpha, H_2^r(\mathbb{R}^d), \delta) &= \sqrt{\widehat{\lambda}_1\delta^2 + \widehat{\lambda}_2} \\
&= \sqrt{\frac{pc^{\sigma-1}}{r}\left(\Delta^2(c(r-\sigma)-1) + \frac{1}{(1+c\sigma)^{r-1}}\right)} \\
&= \sqrt{\frac{p(\Delta^{-2/r}-1)^{\sigma-1}}{r\sigma^{\sigma-1}}\left(\Delta^2\left(\frac{r-\sigma}{\sigma}(\Delta^{-2/r}-1)-1\right) + \Delta^{2-2/r}\right)} \\
&= \frac{\sqrt{p}}{\sigma^{\sigma/2}}\Delta^{1-\sigma/r}\left(1-\Delta^{2/r}\right)^{\sigma/2}.
\end{aligned}
$$

For $\delta \geq (2\pi)^{d/2}\Delta_0$, taking into account that $\widehat{\lambda}_1 = 0$ and $c = (r-\sigma)^{-1}$, we obtain that

$$
E_2(D^\alpha, H_2^r(\mathbb{R}^d), \delta) = \sqrt{\widehat{\lambda}_2} = \frac{\sqrt{p}}{r^{r/2}}(r-\sigma)^{(r-\sigma)/2}
$$

and the method $\widehat{m}(y) = 0$ is optimal.

Lecture 24

We proved the following theorem.

Theorem 15. *Let* $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d_+$, $\alpha \neq 0$, $r \geq 1$ *and* $\sigma < r$. *If* $0 < \delta < (2\pi)^{d/2}\Delta_0$, *then*

$$E_2(D^\alpha, H_2^r(\mathbb{R}^d), \delta) = \frac{\sqrt{p}}{\sigma^{\sigma/2}}\Delta^{1-\sigma/r}\left(1 - \Delta^{2/r}\right)^{\sigma/2}$$

and the method (65) *is optimal. If* $\delta \geq (2\pi)^{d/2}\Delta_0$, *then*

$$E_2(D^\alpha, H_2^r(\mathbb{R}^d), \delta) = \frac{\sqrt{p}}{r^{r/2}}(r-\sigma)^{(r-\sigma)/2},$$

and the method $\widehat{m}(y) = 0$ *is optimal.*

Now we assume that the Fourier transform of $x \in H_2^r(\mathbb{R}^d)$ is known with an error on some measurable set $\Omega \subset \mathbb{R}^d$. Then we define the error of optimal recovery by

$$E_2(D^\alpha, H_2^r(\mathbb{R}^d), \delta, \Omega) = \inf_{m\colon L_2(\Omega)\to L_2(\mathbb{R}^d)} \sup_{\substack{x\in H_2^r(\mathbb{R}^d),\ y\in L_2(\Omega)\\ \|Fx-y\|_{L_2(\Omega)}\leq\delta}} \|D^\alpha x - m(y)\|_{L_2(\mathbb{R}^d)}.$$

It is easy to verify that for if $\Omega_1 \subset \Omega_2$, then

$$E_2(D^\alpha, H_2^r(\mathbb{R}^d), \delta, \Omega_1) \geq E_2(D^\alpha, H_2^r(\mathbb{R}^d), \delta, \Omega_2).$$

It appears that there exists a set $\Omega_\delta \subset \mathbb{R}^d$ such that for all measurable sets Ω, $\Omega_\delta \subseteq \Omega \subseteq \mathbb{R}^d$, the equality

$$E_2(D^\alpha, H_2^r(\mathbb{R}^d), \delta, \Omega_\delta) = E_2(D^\alpha, H_2^r(\mathbb{R}^d), \delta)$$

holds. In other words, any information about the Fourier transform obtained with the same error outside the set Ω_δ does not lead to decreasing of the error of optimal recovery. Since for $\delta \geq (2\pi)^{d/2}\Delta_0$ we do not use any information (optimal method of recovery is $\widehat{m}(y) = 0$) for such δ, $\Omega_\delta = \emptyset$.

The precise result can be formulated as follows.

Theorem 16. *With the same conditions as in Theorem* 15 *for* $\delta < (2\pi)^{d/2}\Delta_0$ *put*

$$\Omega_\delta = \left\{ t \in \mathbb{R}^d : \frac{|t|^{2\alpha}}{\left(1 + \|t\|^2\right)^r} > \frac{p}{r\sigma^{\sigma-1}}\left(1 - \Delta^{2/r}\right)^{\sigma-1}\Delta^{2(1-\sigma/r)} \right\}.$$

Then for all measurable sets Ω *such that* $\Omega_\delta \subseteq \Omega \subseteq \mathbb{R}^d$

$$E_2(D^\alpha, H_2^r(\mathbb{R}^d), \delta, \Omega) = E_2(D^\alpha, H_2^r(\mathbb{R}^d), \delta),$$

and the method

$$\widehat{m}(y)=\frac{1}{(2\pi)^d}\int_{\Omega_\delta}\frac{(i\tau)^\alpha y(\tau)e^{i\langle\tau,t\rangle}}{1+\dfrac{\sigma\Delta^2}{r(\Delta_0^{2/r}-\Delta^{2/r})}(1+\|\tau\|^2)^r}\,d\tau$$

is optimal.

Lecture 25

Proof. The scheme of the proof is the same as in the previous theorem. We consider the dual extremal problem. Then pass to the Fourier transform and consider the Lagrange function for the extensional extremal problem

$$\mathcal{L}(\mu, \widehat{\lambda}_1, \widehat{\lambda}_2) = \int_{\mathbb{R}^d} \left(-|t|^{2\alpha} + (2\pi)^d \widehat{\lambda}_1 \chi_\Omega(t) + \widehat{\lambda}_2 \left(1 + \|t\|^2\right)^r \right) d\mu(t),$$

where $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$ are defined by (64). It was proved that for all $t \in \mathbb{R}^d$

$$-|t|^{2\alpha} + (2\pi)^d \widehat{\lambda}_1 + \widehat{\lambda}_2 \left(1 + \|t\|^2\right)^r \geq 0.$$

If $t \notin \Omega$, then $t \notin \Omega_\delta$. Consequently,

$$\frac{|t|^{2\alpha}}{\left(1 + \|t\|^2\right)^r} \leq \frac{p}{r\sigma^{\sigma-1}} \left(1 - \Delta^{2/r}\right)^{\sigma-1} \Delta^{2(1-\sigma/r)} = \widehat{\lambda}_2.$$

It means that

$$-|t|^{2\alpha} + \widehat{\lambda}_2 \left(1 + \|t\|^2\right)^r \geq 0.$$

Since

$$-|\widehat{\tau}|^{2\alpha} + (2\pi)^d \widehat{\lambda}_1 + \widehat{\lambda}_2 \left(1 + \|\widehat{\tau}\|^2\right)^r = 0$$

we have

$$-|\widehat{\tau}|^{2\alpha} + \widehat{\lambda}_2 \left(1 + \|\widehat{\tau}\|^2\right)^r = -(2\pi)^d \widehat{\lambda}_1 < 0.$$

Hence $\widehat{\tau} \in \Omega_\delta$. Then the proof proceed exactly in the same way as in the previous theorem. □

Consider the following example. Let $d = 2$, $r = 4$, and $\alpha = (1,1)$. In other words, we consider the problem of optimal recovery of $x''_{t_1 t_2}$ on the class $H_2^4(\mathbb{R}^2)$. It follows from Theorems 15 and 16 that for $0 < \delta < \pi/2$

$$E_2(D^{(1,1)}, H_2^4(\mathbb{R}^2), \delta) = \frac{1}{2\sqrt{2}} \sqrt{\frac{\delta}{\pi}} \left(1 - \sqrt{\frac{\delta}{\pi}}\right),$$

Ω_δ is the set of points $(\rho \sin\varphi, \rho\cos\varphi)$ such that

$$1 + \rho^2 < \left(\frac{\delta}{4\pi} \left(1 - \sqrt{\frac{\delta}{2\pi}}\right)\right)^{-1/4} \rho \sqrt{|\sin 2\varphi|},$$

and the method

$$\widehat{m}(y) = \frac{1}{(2\pi)^4} \int_{\Omega_\delta} \frac{-\tau_1 \tau_2 y(\tau_1, \tau_2) e^{i(\tau_1 t_1 + \tau_2 t_2)}}{1 + \dfrac{\delta^2}{4\pi^2} \left(1 - \sqrt{\frac{\delta}{2\pi}}\right)^{-1} (1 + \tau_1^2 + \tau_2^2)^4} \, d\tau_1 d\tau_2$$

is optimal.

15. Optimal recovery of values of derivatives and Stechkin's problem

We consider optimal recovery problem of $x^{(k)}(\tau)$ where $0 \le k < r$, $\tau \in \mathbb{R}$, on the class $F^r_{2,p}$ by the information about the Fourier transform Fx given on the interval $\Delta_\sigma = (-\sigma, \sigma)$, $0 < \sigma \le \infty$, with the error $\delta > 0$ in the metric $L_p(\Delta_\sigma)$. That is, we would like to find the error of optimal recovery

$$E^\sigma_p(D^k_\tau, F^r_{2,p}, \delta) = \inf_{m\colon L_p(\Delta_\sigma)\to\mathbb{R}} \sup_{\substack{x\in F^r_{2,p},\ y\in L_p(\Delta_\sigma)\\ \|Fx-y\|_{L_p(\Delta_\sigma)}\le\delta}} |x^{(k)}(\tau) - m(y)|$$

and an optimal method of recovery.

We also study the problem of best approximation of $x^{(k)}(\tau)$, $0 \le k < r$, $\tau \in \mathbb{R}$, on the class $F^r_{2,p}$ by the information about the Fourier transform Fx given on the interval Δ_σ by means of linear continuous functionals on $L_p(\Delta_\sigma)$ with the norm not greater than some fixed positive number N. It is in finding the value

$$S^\sigma_p(D^k_\tau, F^r_{2,p}, N) = \inf_{y^*} \sup_{x\in F^r_{2,p}} |x^{(k)}(\tau) - \langle y^*, Fx\rangle| \tag{66}$$

(where the lower bound is taken over all linear functionals y^* on $L_p(\Delta_\sigma)$ such that $\|y^*\| \le N$), and also a functional $\widehat{y}^*$ delivering the lower bound in (66) which is called *extremal*.

If we put x in (66) instead of Fx then we obtain the classical problem of S. B. Stechkin, so (66) we also call the problem of Stechkin.

In view of the translation invariance of the classes under consideration throughout what follows we assume that $\tau = 0$.

Lecture 26

Theorem 17. *Let $r \in \mathbb{N}$, $k \in \mathbb{Z}_+$, $0 \le k < r$, $0 < \sigma \le \infty$, $\delta > 0$, $1 \le p \le \infty$, and for all $x \in \mathcal{F}^r_{2,p}$ the equality*

$$x^{(k)}(0) = \langle \widehat{y}^*, Fx \rangle + \lambda \int_{\mathbb{R}} x^{(r)}(t)\overline{\widehat{x}^{(r)}(t)}\, dt \tag{67}$$

holds, where $\widehat{y}^*$ is some linear continuous functional on $L_p(\Delta_\sigma)$, $\lambda \in \mathbb{R}_+$, and $\widehat{x} \in \mathcal{F}^r_{2,p}$ satisfies the following conditions

(*i*) $\|F\widehat{x}\|_{L_p(\Delta_\sigma)} = \delta$,
(*ii*) $\|\widehat{x}^{(r)}\|_{L_2(\mathbb{R})} = 1$,
(*iii*) $\langle \widehat{y}^*, F\widehat{x} \rangle = \delta \|\widehat{y}^*\|$.

Then

$$E^\sigma_p(D^k_0, F^r_{2,p}, \delta) = \sup_{\substack{x \in F^r_{2,p},\\ \|Fx\|_{L_p(\Delta_\sigma)} \le \delta}} |x^{(k)}(0)| = \lambda + \delta \|\widehat{y}^*\| \tag{68}$$

and $\widehat{y}^*$ is an optimal method of recovery. Moreover, for Stechkin's problem for $N = \|\widehat{y}^*\|$

$$S^\sigma_p(D^k_0, F^r_{2,p}, N) = \lambda$$

and $\widehat{y}^*$ is an extremal functional.

Proof. It follows from (67) that for all $x \in F^r_{2,p}$

$$|x^{(k)}(0) - \langle \widehat{y}^*, Fx \rangle| \le \lambda \|x^{(r)}\|_{L_2(\mathbb{R})} \|\widehat{x}^{(r)}\|_{L_2(\mathbb{R})} \le \lambda.$$

Thus,

$$\begin{aligned} E^\sigma_p(D^k_0, F^r_{2,p}, \delta) &\le \sup_{\substack{x \in F^r_{2,p},\ y \in L_p(\Delta_\sigma)\\ \|Fx - y\|_{L_p(\Delta_\sigma)} \le \delta}} |x^{(k)}(0) - \langle \widehat{y}^*, y \rangle| \\ &\le \sup_{\substack{x \in F^r_{2,p},\ y \in L_p(\Delta_\sigma)\\ \|Fx - y\|_{L_p(\Delta_\sigma)} \le \delta}} \left(|x^{(k)}(0) - \langle \widehat{y}^*, Fx \rangle| + |\langle \widehat{y}^*, Fx - y \rangle|\right) \\ &\le \sup_{x \in F^r_{2,p}} |x^{(k)}(0) - \langle \widehat{y}^*, Fx \rangle| + \delta \|\widehat{y}^*\| = \lambda + \delta \|\widehat{y}^*\|. \end{aligned} \tag{69}$$

On the other hand, using the general result about the lower bound (see Lemma 2) and taking (*ii*) and (*iii*) into account we have

$$\begin{aligned} E^\sigma_p(D^k_0, F^r_{2,p}, \delta) &\ge \sup_{\substack{x \in F^r_{2,p},\\ \|Fx\|_{L_p(\Delta_\sigma)} \le \delta}} |x^{(k)}(0)| \ge |\widehat{x}^{(k)}(0)| \\ &= \left|\langle \widehat{y}^*, F\widehat{x} \rangle + \lambda \|\widehat{x}^{(r)}\|_{L_2(\mathbb{R})}\right| = \lambda + \delta \|\widehat{y}^*\|. \end{aligned}$$

It follows from this inequality and (69) equality (68) and the optimality of the method $\widehat{y}^*$.

We now proceed to the Stechkin problem. As was proved, there exists an optimal method of recovery defined by a linear continuous functional, therefore

$$
\begin{aligned}
E_p^\sigma(D_0^k, F_{2,p}^r, \delta) &= \inf_{N>0} \inf_{\|y^*\| \le N} \sup_{\substack{x \in F_{2,p}^r,\ y \in L_p(\Delta_\sigma) \\ \|Fx - y\|_{L_p(\Delta_\sigma)} \le \delta}} |x^{(k)}(0) - \langle y^*, y\rangle| \\
&\le \inf_{\|y^*\| \le N} \sup_{\substack{x \in F_{2,p}^r,\ y \in L_p(\Delta_\sigma) \\ \|Fx - y\|_{L_p(\Delta_\sigma)} \le \delta}} (|x^{(k)}(0) - \langle y^*, Fx\rangle| + \langle y^*, Fx - y\rangle|) \\
&\le \inf_{\|y^*\| \le N} \sup_{x \in F_{2,p}^r} |x^{(k)}(0) - \langle y^*, Fx\rangle| + \delta N = S_p^\sigma(D_\tau^k, F_{2,p}^r, N) + \delta N.
\end{aligned}
$$

Consequently, for all $N > 0$

$$
S_p^\sigma(D_\tau^k, F_{2,p}^r, N) \ge E_p^\sigma(D_0^k, F_{2,p}^r, \delta) - \delta N. \tag{70}
$$

Hence from (68) for $N = \|\widehat{y}^*\|$ we obtain

$$
S_p^\sigma(D_\tau^k, F_{2,p}^r, N) \ge \lambda.
$$

On the other hand, in view of (67) we have

$$
S_p^\sigma(D_\tau^k, F_{2,p}^r, N) \le \sup_{x \in F_{2,p}^r} |x^{(k)}(0) - \langle \widehat{y}^*, Fx\rangle| = \lambda.
$$

□

In view of the translation invariance of the space $\mathcal{F}_{2,p}^r$ it follows from Corollary 2 and (68) the following result.

Corollary 5. *Assume that the conditions of Theorem* 17 *are fulfilled for $\sigma = \infty$. Then*

$$
K_F(k, r, \infty, p, 2) = \lambda + \|y^*\|.
$$

Corollary 5 states that if the conditions of Theorem 17 are fulfilled for $\sigma = \infty$, then the exact inequality for derivatives has the following form

$$
\|x^{(k)}\|_{L_\infty(\mathbb{R})} \le (\lambda + \|y^*\|) \|Fx\|_{L_p(\mathbb{R})}^{\frac{r-k-1/2}{r+1/p'-1/2}} \|x^{(r)}\|_{L_2(\mathbb{R}^d))}^{\frac{k+1/p'}{r+1/p'-1/2}}. \tag{71}
$$

Lecture 27

We start with the case when $p = \infty$.

Theorem 18. *Let $\delta > 0$, $k, r \in \mathbb{Z}$, $0 \le k < r$, $0 < \sigma \le \infty$,*

$$\widehat{\sigma} = \left(\frac{\pi(2r+1)(2r-2k-1)}{2\delta^2(2r-k)}\right)^{\frac{1}{2r+1}},$$

and $\sigma_0 = \min(\sigma, \widehat{\sigma})$. Then

$$E_\infty^\sigma(D_0^k, F_{2,\infty}^r, \delta) = \frac{\sigma_0^{k+1}}{\pi}\left(\frac{\delta}{k+1} + \sqrt{\frac{1}{2r-2k-1}\left(\frac{\pi}{\sigma_0^{2r+1}} - \frac{\delta^2}{2r+1}\right)}\right)$$

and the method

$$\widehat{m}(y) = \frac{1}{2\pi}\int_{|t|<\sigma_0} (it)^k\left(1 - \delta\lambda|t|^{2r-k}\right) y(t)\,dt,$$

where

$$\lambda = \frac{\sigma_0^{-2r+k}}{\sqrt{2r-2k-1}}\left(\frac{\pi}{\sigma_0^{2r+1}} - \frac{\delta^2}{2r+1}\right)^{-1/2},$$

is optimal.

Proof. Let us prove that for all $x \in \mathcal{F}_{2,\infty}^r$ the equality

$$x^{(k)}(0) = \frac{1}{2\pi}\int_{|t|<\sigma_0} (it)^k\left(1 - \delta\lambda|t|^{2r-k}\right) Fx(t)\,dt + \lambda\int_{\mathbb{R}} x^{(r)}(t)\overline{\widehat{x}^{(r)}(t)}\,dt \tag{72}$$

holds, where the function $\widehat{x} \in \mathcal{F}_{2,\infty}^r$ is such that

$$F\widehat{x}(t) = \begin{cases} (-i)^k\delta\,\mathrm{sign}\,t^k, & |t| < \sigma_0, \\ \dfrac{(-i)^k}{\lambda t^{2r-k}}, & |t| \ge \sigma_0. \end{cases}$$

By the Plancherel theorem we have

$$\int_{\mathbb{R}} x^{(r)}(t)\overline{\widehat{x}^{(r)}(t)}\,dt = \frac{1}{2\pi}\int_{\mathbb{R}} t^{2r} Fx(t)\overline{F\widehat{x}(t)}\,dt.$$

Therefore,

$$\frac{1}{2\pi}\int_{|t|<\sigma_0}(it)^k\left(1-\delta\lambda|t|^{2r-k}\right)Fx(t)\,dt+\lambda\int_{\mathbb{R}}x^{(r)}(t)\overline{\widehat{x}^{(r)}(t)}\,dt$$
$$=\frac{1}{2\pi}\int_{|t|<\sigma_0}\left((it)^k\left(1-\delta\lambda|t|^{2r-k}\right)+\lambda t^{2r}i^k\delta\operatorname{sign}t^k\right)Fx(t)\,dt$$
$$+\frac{1}{2\pi}\int_{|t|\ge\sigma_0}(it)^kFx(t)\,dt=\frac{1}{2\pi}\int_{\mathbb{R}}(it)^kFx(t)\,dt=x^{(k)}(0).$$

The equality $\|\widehat{x}^{(r)}\|_{L_2(\mathbb{R})}=1$ is easily verified. Let us prove that $\|F\widehat{x}\|_{L_\infty(\Delta_\sigma)}=\delta$. For $\sigma_0\ge\sigma$ it is immediately follows from the definition of $F\widehat{x}$. Let $\sigma_0<\sigma$. Then $\sigma_0=\widehat{\sigma}$ and it is not difficult to verify that $(\lambda\widehat{\sigma}^{2r-k})^{-1}=\delta$. Thus, $|F\widehat{x}(t)|\le\delta$ for $|t|\ge\widehat{\sigma}$. We now verify the fulfilment of the condition (iii) of Theorem 7. We have

$$\langle\widehat{y}^*,F\widehat{x}\rangle=\frac{\delta}{2\pi}\int_{|t|<\sigma_0}|t|^k\left(1-\delta\lambda|t|^{2r-k}\right)\,dt. \tag{73}$$

Let us prove that $1-\delta\lambda|t|^{2r-k}>0$ for $|t|<\sigma_0$. In view of the definition of σ_0 we have

$$\delta^2\sigma_0^{2r+1}2(2r-k)\le\delta^2\widehat{\sigma}^{2r+1}2(2r-k)=\pi(2r+1)(2r-2k-1).$$

Hence

$$\delta^2\sigma_0^{2r+1}(2r+1)\le(2r-2k-1)(\pi(2r+1)-\delta^2\sigma_0^{2r+1})$$
$$=\sigma_0^{-2r+2k+1}(2r+1)\lambda^{-2},$$

that is, $\delta\lambda\sigma_0^{2r-k}\le1$. Thus, for $|t|<\sigma_0$, $1-\delta\lambda|t|^{2r-k}>1-\delta\lambda\sigma_0^{2r-k}\ge0$. Consequently, the right-hand side of (73) is equal to $\delta\|\widehat{y}^*\|$. To complete the proof it remains to apply Theorem 7. □

It follows by Theorem 18 that for $\sigma\ge\widehat{\sigma}$

$$E_\infty^\sigma(D_0^k,F_{2,\infty}^r,\delta)=K\delta^{\frac{2r-2k-1}{2r+1}},$$

where

$$K=\frac{(r+1/2)^{\frac{k+1}{2r+1}}}{k+1}\left(\frac{2r-k}{\pi(2r-2k-1)}\right)^{\frac{2r-k}{2r+1}}. \tag{74}$$

Thus in the problem under consideration the "saturation" effect of the optimal recovery error is occurred which is in the fact that for a fixed $\delta>0$ the knowledge of the Fourier transform of a function from $F_{2,\infty}^r$ given with the error δ in the uniform metric on the intervals larger than

$(-\widehat{\sigma}, \widehat{\sigma})$ does not result in a decrease in the optimal recovery error. Thus the violation of the relation

$$\delta^2\sigma^{2r+1} \le \frac{\pi(2r+1)(2r-2k-1)}{2(2r-k)}$$

leads to the fact that the available information turns out to be redundant. This fact is apparently important in practical applications when we have to take into account that obtaining the additional information requires some expense.

It follows from (71)

Corollary 6. *Let $k, r \in \mathbb{Z}$ and $0 \le k < r$. Then we have the exact inequality*

$$\|x^{(k)}\|_{L_\infty(\mathbb{R})} \le K\|Fx\|_{L_\infty(\mathbb{R})}^{\frac{2r-2k-1}{2r+1}} \|x^{(r)}\|_{L_2(\mathbb{R})}^{\frac{2k+2}{2r+1}},$$

where the constant K is defined by the equation (74).