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On Optimal Recovery of Heat Equation Solutions*

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We devote this article to Borislav Bojanov who was one of the first mathematicians to undertake the study of optimal recovery problems

In this paper, we consider some optimal recovery problems which are representatives of a vast number of problems in numerical analysis. We focus on the so called *cleaning* phenomenon, where only a part of the given information is used for the construction of an optimal recovery method in the uniform norm.

There are a lot of results concerning the optimal recovery of *linear functionals* (see, for example, [1]–[5] and the references therein). However, the problems of optimal recovery of *linear operators* are not studied that extensively (see [6]–[8]). Here, we present some results about optimal recovery of solutions to differential equations and illustrate our approach in the case of solutions to the heat equation $u_t = u_{xx}$.

1. Periodic Case

We consider the following problem for the heat equation:

$$\begin{gathered} u_t = u_{xx}, \\ u(0,t) = u(\pi,t) = 0, \qquad u(x,0) = f(x). \end{gathered} \tag{1.1}$$

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It is well known that the solution to this problem is given by the series

$$u(x,t) = \sum_{k=1}^{\infty} b_k(f) e^{-k^2 t} \sin kx,$$

with

$$b_k(f) = \frac{2}{\pi} \int_0^{\pi} f(x) \sin kx \, dx.$$

We denote by $\mathcal{W}_2^r[0,\pi]$ the *Sobolev space*

$$\mathcal{W}_2^r[0,\pi] := \big\{ f \in L_2[0,\pi] : \ f^{(r-1)} \text{– abs. cont. on } [0,\pi],\ f^{(r)} \in L_2[0,\pi] \big\},$$

and by $W_2^r[0,\pi]$ the set

$$W_2^r[0,\pi] := \big\{ f \in \mathcal{W}_2^r[0,\pi] : \ \|f^{(r)}\|_{L_2[0,\pi]} \le 1 \big\},$$

where the usual definition of the $L_2[0,\pi]$ norm is

$$\|g\|_{L_2[0,\pi]} = \Big(\frac{2}{\pi} \int_0^{\pi} |g(x)|^2 \, dx \Big)^{1/2}.$$

We are interested in the recovery of the solution to problem (1.1) at some fixed time $t = T$, provided that $u(0,\cdot) = f \in W_2^r[0,\pi]$ and we know with some accuracy δ the vector $b^N(f) = (b_1(f), \ldots, b_N(f))$ of the first N Fourier coefficients of f, namely, a vector $y = (y_1, \ldots, y_N)$ for which $\|b^N(f) - y\|_{\ell_p^N} \le \delta$ is available. Here the ℓ_p^N norm of $a = (a_1, \ldots, a_N)$ is given by

$$\|a\|_{\ell_p^N} = \begin{cases} \Big(\sum_{k=1}^{N} |a_k|^p \Big)^{1/p}, & 1 \le p < \infty \\ \max_{1 \le k \le N} |a_k|, & p = \infty. \end{cases}$$

This type of information is denoted by $\mathrm{Four}_{N,\delta,p}$, and the corresponding recovery problem is denoted by $\mathcal{R}(u(\cdot,T), W_2^r[0,\pi], \mathrm{Four}_{N,\delta,p})$. An arbitrary mapping $\varphi : \mathbb{R}^N \to L_2[0,\pi]$ generates a *recovery method*, the value

$$e(\mathcal{R},\varphi) = \sup_{f \in W_2^r[0,\pi]} \ \sup_{\substack{y \in \mathbb{R}^N \\ \|b^N(f) - y\|_{\ell_p^N} \le \delta}} \|u(\cdot,T) - \varphi(y)\|_{L_2[0,\pi]}$$

is called *the error of the method* φ, the value

$$E(\mathcal{R}) = \inf_{\varphi : \mathbb{R}^N \to L_2[0,\pi]} e(\mathcal{R},\varphi)$$

is called *the error of the* $\mathcal{R}$*-recovery problem*, and a method for which the infimum is attained is called *an optimal method*.

1.1. Case $p = 2$

We denote by $\mathcal{R}_1$ the recovery problem in the case $p = 2$. The following theorem is true.

Theorem 1. *For all* $0 < \delta < 1$, *the error of the recovery problem* $\mathcal{R}_1$ *is*

$$E(\mathcal{R}_1) = e^{-T}\sqrt{\delta^2 + \frac{1-\delta^2}{(N+1)^{2r}}\, e^{-2TN(N+2)}},$$

and the method

$$u(x,T) \approx \sum_{k=1}^{N}\Big(1 + \frac{k^{2r}}{(N+1)^{2r}e^{2TN(N+2)} - 1}\Big)^{-1} y_k e^{-k^2T}\sin kx$$

is optimal. For $\delta \geq 1$,

$$E(\mathcal{R}_1) = e^{-T},$$

and $u(x,T) \approx 0$ *is an optimal method.*

Proof. From general results on recovery problems (see, for example [7, Lemma 1]), one can obtain the lower bound

$$E(\mathcal{R}_1) \geq \sup_{\substack{f\in W_2^r[0,\pi]\\ \|b^N(f)\|_{\ell_2^N}\leq\delta}} \|u(\cdot,T)\|_{L_2[0,\pi]}. \tag{1.2}$$

Using the Parseval's identity, the extremal problem in the right hand-side of (1.2) (with $\|u(\cdot,T)\|_{L_2[0,\pi]}$ replaced by $\|u(\cdot,T)\|^2_{L_2[0,\pi]}$) can be rewritten as

$$\sum_{k=1}^{\infty} b_k^2(f)e^{-2k^2T} \to \max, \qquad \sum_{k=1}^{N} b_k^2(f) \leq \delta^2, \quad \sum_{k=1}^{\infty} b_k^2(f)k^{2r} \leq 1. \tag{1.3}$$

We set $u_k = b_k^2(f)$, write (1.3) in the form

$$\sum_{k=1}^{\infty} u_k e^{-2k^2T} \to \max, \qquad \sum_{k=1}^{N} u_k \leq \delta^2, \quad \sum_{k=1}^{\infty} u_k k^{2r} \leq 1, \quad u_k \geq 0, \tag{1.4}$$

and consider the Lagrange function of (1.4):

$$\mathcal{L}(\{u_k\}_1^\infty, \lambda_1, \lambda_2) := \sum_{k=1}^{\infty}\big(-e^{-2k^2T} + \lambda_1\chi_k + \lambda_2 k^{2r}\big)u_k,$$

where

$$\chi_k = \begin{cases} 1, & 1 \leq k \leq N \\ 0, & k > N. \end{cases}$$

It is easy to check that if there exists an admissible (in (1.4)) sequence $\{\widehat{u}_k\}_1^\infty$ and numbers $\widehat{\lambda}_1, \widehat{\lambda}_2 \geq 0$, such that

$$\min_{u_k \geq 0} \mathcal{L}(\{u_k\}_1^\infty, \widehat{\lambda}_1, \widehat{\lambda}_2) = \mathcal{L}(\{\widehat{u}_k\}_1^\infty, \widehat{\lambda}_1, \widehat{\lambda}_2), \tag{1.5}$$

and if the conditions of complementary slackness

$$\widehat{\lambda}_1\Big(\sum_{k=1}^N \widehat{u}_k - \delta^2\Big) + \widehat{\lambda}_2\Big(\sum_{k=1}^\infty \widehat{u}_k k^{2r} - 1\Big) = 0, \tag{1.6}$$

are satisfied, then $\{\widehat{u}_k\}_1^\infty$ is a solution to (1.4).

For $0 < \delta < 1$, we choose

$$\widehat{u}_1 = \delta^2, \qquad \widehat{u}_{N+1} = \frac{1-\delta^2}{(N+1)^{2r}}, \qquad \widehat{u}_k = 0, \quad k \neq 1, N+1,$$
$$\widehat{\lambda}_1 = e^{-2T} - \frac{e^{-2(N+1)^2 T}}{(N+1)^{2r}}, \qquad \widehat{\lambda}_2 = \frac{e^{-2(N+1)^2 T}}{(N+1)^{2r}},$$

and for $\delta \geq 1$, we select

$$\widehat{u}_1 = 1, \qquad \widehat{u}_k = 0, \quad k = 2, 3, \dots,$$
$$\widehat{\lambda}_1 = 0, \qquad \widehat{\lambda}_2 = e^{-2T}.$$

It is clear that for such $\{\widehat{u}_k\}_1^\infty$ and $\widehat{\lambda}_1, \widehat{\lambda}_2$, (1.5) and (1.6) hold. Consequently, the extremal value to problem (1.4) is

$$e^{-2T}\Big(\delta^2 + \frac{1-\delta^2}{(N+1)^{2r}} e^{-2TN(N+2)}\Big),$$

for $0 < \delta < 1$, and e^{-2T} for $\delta \geq 1$.

Note that analogous arguments show that $\{\widehat{u}_k\}_1^\infty$ is a solution to

$$\sum_{k=1}^\infty u_k e^{-2k^2 T} \to \max, \qquad \widehat{\lambda}_1 \sum_{k=1}^N u_k + \widehat{\lambda}_2 \sum_{k=1}^\infty u_k k^{2r} \leq \widehat{\lambda}_1 \delta^2 + \widehat{\lambda}_2, \quad u_k \geq 0.$$

Next, we construct an optimal method of recovery. We consider the following extremal problem for a given $y \in \mathbb{R}^N$:

$$\widehat{\lambda}_1 \|b^N(f) - y\|^2_{\ell_2^N} + \widehat{\lambda}_2 \|f^{(r)}\|^2_{L_2[0,\pi]} \to \min, \qquad f \in \mathcal{W}_2^r[0,\pi]. \tag{1.7}$$

Direct calculations show that the function

$$\widehat{f}(x) = \sum_{k=1}^N \frac{\widehat{\lambda}_1}{\widehat{\lambda}_1 + \widehat{\lambda}_2 k^{2r}}\, y_k \sin kx \tag{1.8}$$

is a solution to (1.7), and hence for all $f \in \mathcal{W}_2^r[0,\pi]$ the following identity holds:

$$\widehat{\lambda}_1\|b^N(f) - b^N(\widehat{f})\|^2_{\ell_2^N} + \widehat{\lambda}_2\|f^{(r)} - \widehat{f}^{(r)}\|^2_{L_2[0,\pi]} + \widehat{\lambda}_1\|b^N(\widehat{f}) - y\|^2_{\ell_2^N} + \widehat{\lambda}_2\|\widehat{f}^{(r)}\|^2_{L_2[0,\pi]} = \widehat{\lambda}_1\|b^N(f) - y\|^2_{\ell_2^N} + \widehat{\lambda}_2\|f^{(r)}\|^2_{L_2[0,\pi]}.$$

If $f \in W_2^r[0,\pi]$, $\|b^N(f) - y\|_{\ell_2^N} \le \delta$, $g := f - \widehat{f}$, we obtain

$$\widehat{\lambda}_1\|b^N(g)\|^2_{\ell_2^N} + \widehat{\lambda}_2\|g^{(r)}\|^2_{L_2[0,\pi]} \le \widehat{\lambda}_1\|b^N(f) - y\|^2_{\ell_2^N} + \widehat{\lambda}_2\|f^{(r)}\|^2_{L_2[0,\pi]} \le \widehat{\lambda}_1\delta^2 + \widehat{\lambda}_2.$$

At the same time, the error of the method

$$u(x,T) \approx \sum_{k=1}^N b_k(\widehat{f})e^{-k^2T}\sin kx$$

is estimated by

$$\Big\|u(x,T) - \sum_{k=1}^N b_k(\widehat{f})e^{-k^2T}\sin kx\Big\|^2_{L_2[0,\pi]} = \sum_{k=1}^\infty b_k^2(g)e^{-2k^2T}$$
$$\le \sup\Big\{\sum_{k=1}^\infty u_k e^{-2k^2T} : \widehat{\lambda}_1\sum_{k=1}^N u_k + \widehat{\lambda}_2\sum_{k=1}^\infty u_k k^{2r} \le \widehat{\lambda}_1\delta^2 + \widehat{\lambda}_2,\ u_k \ge 0\Big\}.$$

Since this supremum coincides with the minimum value of problem (1.4), the estimates from above and from below are equal, and therefore this method is optimal. Substituting $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$ in (1.8) gives the required result. □

1.2. Case $p = \infty$

We denote by $\mathcal{R}_2$ the recovery problem for $p = \infty$. The following theorem holds.

Theorem 2. *If* $m := \max\Big\{n \in \mathbb{Z}_+ : \delta^2\sum_{k=1}^n k^{2r} < 1,\ 0 \le n \le N\Big\}$, *then the error of the recovery problem* $\mathcal{R}_2$ *is*

$$E(\mathcal{R}_2) = \sqrt{\delta^2\sum_{k=1}^m \alpha_k e^{-2k^2T} + e^{-2(m+1)^2T}(m+1)^{-2r}},$$

where $\alpha_k := 1 - \big(k/(m+1)\big)^{2r}e^{-2T(m+k+1)(m-k+1)}$, $k = 1,\dots,m$. *The method*

$$u(x,T) \approx \sum_{k=1}^m \alpha_k y_k e^{-k^2T}\sin kx$$

is optimal.

Proof. As in (1.2), we have

$$E(\mathcal{R}_2) \geq \sup_{\substack{f\in W_2^r[0,\pi] \\ \|b^N(f)\|_{\ell_\infty^N}\leq\delta}} \|u(\cdot,T)\|_{L_2[0,\pi]}.$$

Similarly to the proof of Theorem 1, we rewrite the extremal problem in the right hand-side of this inequality in the form

$$\sum_{k=1}^{\infty} u_k e^{-2k^2T} \to \max, \quad 0\leq u_k\leq\delta^2,\ k=1,\ldots,N,\ \sum_{k=1}^{\infty} u_k k^{2r}\leq 1, \qquad (1.9)$$

where $u_k = b_k^2(f)$, and we consider the Lagrange function of (1.9)

$$\mathcal{L}(\{u_k\}_1^\infty,\lambda) := \sum_{k=1}^{\infty}\left(-e^{-2k^2T}+\lambda_{N+1}k^{2r}\right)u_k + \sum_{k=1}^{N}\lambda_k u_k,$$

$\lambda := (\lambda_1,\ldots,\lambda_{N+1})$. To solve problem (1.9), it is sufficient to find an admissible sequence $\{\widehat{u}_k\}_1^\infty$ and a vector $\widehat{\lambda}\geq 0$, such that

$$\min_{u_k\geq 0}\mathcal{L}(\{u_k\}_1^\infty,\widehat{\lambda}) = \mathcal{L}(\{\widehat{u}_k\}_1^\infty,\widehat{\lambda}) \qquad (1.10)$$

and

$$\sum_{k=1}^{N}\widehat{\lambda}_k(\widehat{u}_k-\delta^2) + \widehat{\lambda}_{N+1}\Big(\sum_{k=1}^{\infty}\widehat{u}_k k^{2r}-1\Big) = 0. \qquad (1.11)$$

Then $\{\widehat{u}_k\}_1^\infty$ will be a solution to (1.9). Let

$$\widehat{\lambda}_{N+1} = (m+1)^{-2r}e^{-2(m+1)^2T},$$

$$\widehat{\lambda}_k = \begin{cases} e^{-2k^2T}-\widehat{\lambda}_{N+1}k^{2r}, & 1\leq k\leq m \\ 0, & m+1\leq k\leq N, \end{cases}$$

and let us define the sequence $\{\widehat{u}_k\}_1^\infty$ as follows:

$$\widehat{u}_k = \begin{cases} \delta^2, & 1\leq k\leq m, \\ \Big(1-\delta^2\sum_{k=1}^{m}k^{2r}\Big)(m+1)^{-2r}, & k=m+1, \\ 0, & k>m+1. \end{cases}$$

If follows from the definition of m that $\{\widehat{u}_k\}_1^\infty$ is an admissible sequence. Moreover,

$$\mathcal{L}(\{u_k\}_1^\infty,\widehat{\lambda}) = \sum_{k=m+2}^{\infty}\left(-e^{-2k^2T}+\widehat{\lambda}_{N+1}k^{2r}\right)u_k \geq 0 = \mathcal{L}(\{\widehat{u}_k\}_1^\infty,\widehat{\lambda}), \quad u_k\geq 0,$$

and thus condition (1.10) is satisfied. One can verify that (1.11) is also satisfied, and therefore $\{\widehat{u}_k\}_1^\infty$ is a solution to (1.9). Hence,

$$E(\mathcal{R}_2) \geq \sqrt{\sum_{k=1}^{\infty} e^{-2k^2T}\widehat{u}_k} = \sqrt{\delta^2 \sum_{k=1}^{m} \alpha_k e^{-2k^2T} + e^{-2(m+1)^2T}(m+1)^{-2r}}.$$

Likewise, one can prove that $\{\widehat{u}_k\}_1^\infty$ is a solution to the problem

$$\sum_{k=1}^{\infty} u_k e^{-2k^2T} \to \max, \quad \sum_{k=1}^{N} \widehat{\lambda}_k u_k + \widehat{\lambda}_{N+1} \sum_{k=1}^{\infty} u_k k^{2r} \leq \delta^2 \sum_{k=1}^{N} \widehat{\lambda}_k + \widehat{\lambda}_{N+1}, \ \ u_k \geq 0.$$

Next, we construct an optimal method of recovery. For every $y \in \mathbb{R}^N$, we consider the extremal problem

$$\sum_{k=1}^{N} \widehat{\lambda}_k |b_k(f) - y_k|^2 + \widehat{\lambda}_{N+1} \|f^{(r)}\|^2_{L_2[0,\pi]} \to \min, \qquad f \in \mathcal{W}_2^r[0,\pi]. \tag{1.12}$$

It is easy to show that the function

$$\widehat{f}(x) = \sum_{k=1}^{m} \frac{\widehat{\lambda}_k}{\widehat{\lambda}_k + \widehat{\lambda}_{N+1}k^{2r}}\, y_k \sin kx \tag{1.13}$$

is a solution to (1.12), and hence for each $f \in \mathcal{W}_2^r[0,\pi]$ the identity

$$\begin{aligned}\sum_{k=1}^{N} \widehat{\lambda}_k |b_k(f) - b_k(\widehat{f})|^2 + \widehat{\lambda}_{N+1} \|f^{(r)} - \widehat{f}^{(r)}\|^2_{L_2[0,\pi]} + \sum_{k=1}^{N} \widehat{\lambda}_k |b_k(\widehat{f}) - y_k|^2 \\ + \widehat{\lambda}_{N+1} \|\widehat{f}^{(r)}\|^2_{L_2[0,\pi]} = \sum_{k=1}^{N} \widehat{\lambda}_k |b_k(f) - y_k|^2 + \widehat{\lambda}_{N+1} \|f^{(r)}\|^2_{L_2[0,\pi]}\end{aligned} \tag{1.14}$$

holds. If $f \in W_2^r[0,\pi]$ and $|b_k(f) - y_k| \leq \delta$, $k = 1, \ldots, N$, then from (1.14) for $g = f - \widehat{f}$, we obtain

$$\begin{aligned}\sum_{k=1}^{N} \widehat{\lambda}_k |b_k(g)|^2 + \widehat{\lambda}_{N+1} \|g^{(r)}\|^2_{L_2[0,\pi]} &\leq \sum_{k=1}^{N} \widehat{\lambda}_k |b_k(f) - y_k|^2 + \widehat{\lambda}_{N+1} \|f^{(r)}\|^2_{L_2[0,\pi]} \\ &\leq \delta^2 \sum_{k=1}^{N} \widehat{\lambda}_k + \widehat{\lambda}_{N+1}.\end{aligned}$$

For the error of the method

$$u(x,T) \approx \sum_{k=1}^{N} b_k(\widehat{f}) e^{-k^2T} \sin kx, \tag{1.15}$$

we have

$$\Big\|u(x,T)-\sum_{k=1}^{N}b_k(\widehat{f})e^{-k^2T}\sin kx\Big\|^2_{L_2[0,\pi]}=\sum_{k=1}^{\infty}b_k^2(g)e^{-2k^2T}$$
$$\le \sup\Big\{\sum_{k=1}^{\infty}u_ke^{-2k^2T}:\sum_{k=1}^{N}\widehat{\lambda}_ku_k+\widehat{\lambda}_{N+1}\sum_{k=1}^{\infty}u_kk^{2r}\le\delta^2\sum_{k=1}^{N}\widehat{\lambda}_k+\widehat{\lambda}_{N+1},\ u_k\ge 0\Big\}.$$

Since this supremum coincides with the minimum value of problem (1.12), the estimate from above is equal to the estimate from below, and hence (1.15) is an optimal method of recovery. Substituting $\widehat{\lambda}_1,\dots,\widehat{\lambda}_{N+1}$ in the definition (1.13) of $\widehat{f}$ gives the required result. □

Let us set

$$\delta_n:=\Big(\sum_{k=1}^{n}k^{2r}\Big)^{-1/2}.$$

If $\delta_{n+1}\le\delta<\delta_n$, Theorem 2 gives that for all $k>n$ the error of the recovery problem $\mathcal{R}(u(\cdot,T),W_2^r[0,\pi],\mathrm{Four}_{k,\delta,\infty})$ is the same as the error of $\mathcal{R}(u(\cdot,T),W_2^r[0,\pi],\mathrm{Four}_{n,\delta,\infty})$. Therefore, if δ is fixed and $\delta_{n+1}\le\delta<\delta_n$, knowing more Fourier coefficients with the same accuracy δ does not decrease the error of optimal recovery.

2. Non-periodic Case

Now, we consider the problem of recovery of the solution to the problem

$$\begin{aligned}u_t&=u_{xx},\\ u(x,0)&=f(x),\quad x\in\mathbb{R},\end{aligned}\tag{2.1}$$

at time $t=T$, knowing the Fourier transform Ff of f on the interval $\Delta_\sigma:=(-\sigma,\sigma)$ with accuracy δ in the $L_2(\Delta_\sigma)$-norm. Similarly to the periodic case, we denote by $\mathcal{W}_2^r(\mathbb{R})$ the *Sobolev space*

$$\mathcal{W}_2^r(\mathbb{R})=\big\{f\in L_2(\mathbb{R}):f^{(r-1)}-\text{loc. abs. cont. on }\mathbb{R},\ \|f^{(r)}\|_{L_2(\mathbb{R})}<\infty\big\},$$

and by $W_2^r(\mathbb{R})$ the set

$$W_2^r(\mathbb{R})=\big\{f\in\mathcal{W}_2^r(\mathbb{R}):\|f^{(r)}\|_{L_2(\mathbb{R})}\le 1\big\},$$

where

$$\|g\|_{L_2(\mathbb{R})}=\Big(\int_{\mathbb{R}}|g(x)|^2\,dx\Big)^{1/2}.$$

2.1. Case $p = 2$

We denote by $\mathcal{R}_3$ the recovery problem $\mathcal{R}(u(\cdot,T), W_2^r(\mathbb{R}), \mathrm{Four}_{\sigma,\delta,2})$ of finding the value

$$E(\mathcal{R}_3) = \inf_{\varphi: L_2(\Delta_\sigma)\to L_2(\mathbb{R})} \sup_{f\in W_2^r(\mathbb{R})} \sup_{\substack{y\in L_2(\Delta_\sigma)\\ \|Ff-y\|_{L_2(\Delta_\sigma)}\le\delta}} \|u(\cdot,T)-\varphi(y)\|_{L_2(\mathbb{R})}.$$

The following theorem is true.

Theorem 3. *The error of the recovery problem $\mathcal{R}_3$ is*

$$E(\mathcal{R}_3) = \sqrt{\frac{\delta^2}{2\pi} + \sigma^{-2r}e^{-2\sigma^2 T}}, \qquad \sigma > 0,$$

and

$$u(x,T) \approx \widehat{m}(y) := \frac{1}{2\pi}\int_{\Delta_\sigma} e^{-\lambda^2 T}\big(1+\sigma^{-2r}e^{-2\sigma^2 T}\lambda^{2r}\big)^{-1} y(\lambda)e^{i\lambda x}\,d\lambda \tag{2.2}$$

is an optimal method.

Proof. Similarly to (1.2), we have

$$E(\mathcal{R}_3) \ge \sup_{\substack{f\in W_2^r(\mathbb{R})\\ \|Ff\|_{L_2(\Delta_\sigma)}\le\delta}} \|u(\cdot,T)\|_{L_2(\mathbb{R})}. \tag{2.3}$$

Using Plancherel's theorem and the fact that $Fu(\cdot,T)(\lambda) = e^{-\lambda^2 T}Ff(\lambda)$, (see, for example, [9, p. 406]), the extremal problem in the right-hand side of (2.3) can be rewritten in the form (for convenience we consider squares)

$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{-2\lambda^2 T}|Ff(\lambda)|^2\,d\lambda \to \max, \quad \int_{\Delta_\sigma} |Ff(\lambda)|^2\,d\lambda \le \delta^2,$$
$$\frac{1}{2\pi}\int_{\mathbb{R}} \lambda^{2r}|Ff(\lambda)|^2\,d\lambda \le 1. \tag{2.4}$$

We extend this problem, replacing $(2\pi)^{-1}|Ff(\lambda)|^2\,d\lambda$ by nonnegative measures. Then problem (2.4) can be extended to

$$\int_{\mathbb{R}} e^{-2\lambda^2 T}\,d\mu(\lambda) \to \max, \quad 2\pi\int_{\Delta_\sigma} d\mu(\lambda) \le \delta^2,$$
$$\int_{\mathbb{R}} \lambda^{2r}\,d\mu(\lambda) \le 1, \quad d\mu(\lambda)\ge 0, \tag{2.5}$$

with corresponding Lagrange function

$$\mathcal{L}(d\mu,\lambda_1,\lambda_2) := \int_{\mathbb{R}} \big(-e^{-2\lambda^2 T} + 2\pi\lambda_1\chi_\sigma(\lambda) + \lambda_2\lambda^{2r}\big)\,d\mu(\lambda),$$

where

$$\chi_\sigma(\lambda) = \begin{cases} 1, & \lambda \in \Delta_\sigma \\ 0, & \lambda \notin \Delta_\sigma. \end{cases}$$

It is easy to prove that if there exists a measure $d\widehat{\mu}$, admissible for (2.5), and $\widehat{\lambda}_1, \widehat{\lambda}_2 \geq 0$, such that

$$\min_{d\mu} \mathcal{L}(d\mu, \widehat{\lambda}_1, \widehat{\lambda}_2) = \mathcal{L}(d\widehat{\mu}, \widehat{\lambda}_1, \widehat{\lambda}_2) \tag{2.6}$$

and

$$\widehat{\lambda}_1 \Big(2\pi \int_{\Delta_\sigma} d\widehat{\mu}(\lambda) - \delta^2 \Big) + \widehat{\lambda}_2 \Big(\int_{\mathbb{R}} \lambda^{2r}\, d\widehat{\mu}(\lambda) - 1 \Big) = 0, \tag{2.7}$$

then $d\widehat{\mu}$ is a solution to problem (2.5). We select

$$\widehat{\lambda}_1 = \frac{1}{2\pi}, \qquad \widehat{\lambda}_2 = \sigma^{-2r} e^{-2\sigma^2 T}, \qquad d\widehat{\mu}(\lambda) = \frac{\delta^2}{2\pi}\delta(\lambda) + \sigma^{-2r}\delta(\lambda - \sigma),$$

where δ is the δ-function at zero. One can verify that for these $d\widehat{\mu}$ and $\widehat{\lambda}_1, \widehat{\lambda}_2$, conditions (2.6) and (2.7) are fulfilled. Thus, the solution to (2.5) is

$$\int_{\mathbb{R}} e^{-2\lambda^2 T}\, d\widehat{\mu}(\lambda) = \frac{\delta^2}{2\pi} + \sigma^{-2r} e^{-2\sigma^2 T}. \tag{2.8}$$

It can be shown, approximating δ-functions by corresponding δ-type sequences, that the solution (2.8) is also a solution to problem (2.4). Thus, we have proved that

$$E(\mathcal{R}_3) \geq \sqrt{\frac{\delta^2}{2\pi} + \sigma^{-2r} e^{-2\sigma^2 T}}.$$

Following the same arguments as above, one can prove that the solution to (2.4) is also a solution to the following problem

$$\frac{1}{2\pi} \int_{\mathbb{R}} e^{-2\lambda^2 T} |Ff(\lambda)|^2\, d\lambda \to \max,$$
$$\widehat{\lambda}_1 \int_{\Delta_\sigma} |Ff(\lambda)|^2\, d\lambda + \widehat{\lambda}_2 \frac{1}{2\pi} \int_{\mathbb{R}} \lambda^{2r} |Ff(\lambda)|^2\, d\lambda \leq \widehat{\lambda}_1 \delta^2 + \widehat{\lambda}_2.$$

Now, we construct an optimal method of recovery. For a given $y \in L_2(\Delta_\sigma)$, we consider the extremal problem

$$\widehat{\lambda}_1 \|Ff - y\|^2_{L_2(\Delta_\sigma)} + \widehat{\lambda}_2 \|f^{(r)}\|^2_{L_2(\mathbb{R})} \to \min, \qquad f \in \mathcal{W}_2^r(\mathbb{R}). \tag{2.9}$$

The solution $\widehat{f}$ to this problem is given by

$$F\widehat{f}(\lambda) = \frac{\widehat{\lambda}_1}{\widehat{\lambda}_1 + \widehat{\lambda}_2 (2\pi)^{-1} \lambda^{2r}}\, y(\lambda) = \Big(1 + \sigma^{-2r} e^{-2\sigma^2 T} \lambda^{2r} \Big)^{-1} y(\lambda), \quad |\lambda| < \sigma,$$

and

$$F\widehat{f}(\lambda) = 0, \quad |\lambda| \ge \sigma.$$

Then, for all $f \in \mathcal{W}_2^r(\mathbb{R})$, we have

$$\widehat{\lambda}_1\|Ff - F\widehat{f}\|^2_{L_2(\Delta_\sigma)} + \widehat{\lambda}_2\|f^{(r)} - \widehat{f}^{(r)}\|^2_{L_2(\mathbb{R})} + \widehat{\lambda}_1\|F\widehat{f} - y\|^2_{L_2(\Delta_\sigma)} + \widehat{\lambda}_2\|\widehat{f}^{(r)}\|^2_{L_2(\mathbb{R})} = \widehat{\lambda}_1\|Ff - y\|^2_{L_2(\Delta_\sigma)} + \widehat{\lambda}_2\|f^{(r)}\|^2_{L_2(\mathbb{R})}.$$

If $f \in W_2^r(\mathbb{R})$, $\|Ff - y\|_{L_2(\Delta_\sigma)} \le \delta$, and $g := f - \widehat{f}$ this equality gives

$$\widehat{\lambda}_1\|Fg\|^2_{L_2(\Delta_\sigma)} + \widehat{\lambda}_2\|g^{(r)}\|^2_{L_2(\mathbb{R})} \le \widehat{\lambda}_1\|Ff - y\|^2_{L_2(\Delta_\sigma)} + \widehat{\lambda}_2\|f^{(r)}\|^2_{L_2(\mathbb{R})} \le \widehat{\lambda}_1\delta^2 + \widehat{\lambda}_2.$$

Now, we estimate the error of method (2.2). We have

$$\begin{aligned}\|u(\cdot,T) - \widehat{m}(y)\|^2_{L_2(\mathbb{R})} &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-2\lambda^2 T}|Fg(\lambda)|^2\,d\lambda \\ &\le \sup\Big\{\frac{1}{2\pi}\int_{\mathbb{R}} e^{-2\lambda^2 T}|Ff(\lambda)|^2\,d\lambda : \widehat{\lambda}_1\int_{\Delta_\sigma}|Ff(\lambda)|^2\,d\lambda \\ &\qquad + \widehat{\lambda}_2\frac{1}{2\pi}\int_{\mathbb{R}}\lambda^{2r}|Ff(\lambda)|^2\,d\lambda \le \widehat{\lambda}_1\delta^2 + \widehat{\lambda}_2\Big\}.\end{aligned}$$

Since the supremum coincides with the solution to (2.4), the estimate from above is equal to the estimate from below, and hence (2.2) is an optimal method of recovery. □

2.2. Case $p = \infty$

Next, we consider the same problem of optimal recovery as before, but this time the Fourier transform of f is given with accuracy δ, measured in the $L_\infty(\Delta_\sigma)$-norm. We denote by $W^r_{2\infty}(\mathbb{R})$ the set

$$W^r_{2\infty}(\mathbb{R}) := \{f :\ f^{(r-1)}\text{– loc. abs. cont. on } \mathbb{R},\ \|f^{(r)}\|_{L_2(\mathbb{R})} \le 1, Ff \in L_\infty(\mathbb{R})\}.$$

We are interested in the recovery problem $\mathcal{R}_4 := \mathcal{R}(u(\cdot,T), W^r_{2\infty}(\mathbb{R}), \mathrm{Four}_{\sigma,\delta,\infty})$, that is, in finding the error

$$E(\mathcal{R}_4) = \inf_{\varphi: L_\infty(\Delta_\sigma)\to L_2(\mathbb{R})}\ \sup_{f\in W^r_{2\infty}(\mathbb{R})}\ \sup_{\substack{y\in L_\infty(\Delta_\sigma)\\ \|Ff-y\|_{L_\infty(\Delta_\sigma)}\le\delta}} \|u(\cdot,T) - \varphi(y)\|_{L_2(\mathbb{R})},$$

and in finding an optimal method of recovery. Similarly to the cases considered, one can prove the following theorem.

Theorem 4. *Let $\sigma > 0$, $\delta > 0$, and $\sigma_0 = \min(\sigma, \widehat{\sigma})$ where*

$$\widehat{\sigma} = \Big(\frac{\pi(2r+1)}{\delta^2}\Big)^{1/(2r+1)}.$$

Then the error of the recovery problem $\mathcal{R}_4$ *is*

$$E(\mathcal{R}_4) = \left(\frac{\delta^2}{\pi}\int_0^{\sigma_0} e^{-2\lambda^2 T}\,d\lambda + \frac{e^{-2\sigma^2 T}}{\sigma^{2r}}\Big(1 - \frac{\delta^2\sigma_0^{2r+1}}{\pi(2r+1)}\Big)\right)^{1/2}$$

and

$$u(x,T) \approx \frac{1}{2\pi}\int_{\Delta_{\sigma_0}} e^{-\lambda^2 T}\left(1 - \Big(\frac{\lambda}{\sigma_0}\Big)^{2r} e^{2(\lambda^2-\sigma_0^2)T}\right) y(\lambda) e^{i\lambda x}\,d\lambda$$

is an optimal method.

It follows from this theorem that for $\sigma \geq \widehat{\sigma}$

$$E(\mathcal{R}_4) = \Big(\frac{\delta^2}{\pi}\int_0^{\widehat{\sigma}} e^{-2\lambda^2 T}\,d\lambda\Big)^{1/2}.$$

It means that for a given δ, starting from $\widehat{\sigma}$, further extension of the interval on which the Fourier transform of a function from $W_{2\infty}^r(\mathbb{R})$ is given with accuracy δ in the uniform metric does not result in a decrease in the recovery error. In other words, if the relation $\delta^2\sigma^{2n+1} \leq \pi(2n+1)$ between the input data and the size of the interval on which the data is measured is violated, then the available information turns out to be redundant. This phenomenon of cleaning also appears in problems of optimal recovery of derivatives when inaccurate Fourier transform are available (see [8]).

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