HADAMARD AND SCHWARZ TYPE THEOREMS AND OPTIMAL RECOVERY IN SPACES OF ANALYTIC FUNCTIONS

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Abstract. We prove a general theorem which gives a necessary condition of extremum in the dual optimal recovery problem in terms of inclusion in certain annihilators. Applications of this theorem yield Hadamard and Schwarz type results. We also construct related optimal recovery methods.

1. Introduction

Let $D \subset \mathbb{C}^k$ be a domain, ν be a probability measure on $\bar{D}$ and X be a closed subspace of $L^2(\nu)$. Consider $D_0, \dots, D_n \subset D$ and probability measures $\mu_0, \dots, \mu_n$ on $D_0, \dots, D_n$ respectively. We suppose that $X \subset L^2(\mu_j)$, $j = 0, 1, \dots, n$. We allow one of D_j to coincide with D. In this case we assume that μ_j coincides with ν.

Write $\mathcal{D} = (D_0, \dots, D_n)$, $\mu = (\mu_0, \dots, \mu_n)$, $\delta = (\delta_1, \dots, \delta_n)$, $y = (y_1, \dots, y_n)$.

An optimal recovery problem for this setting is stated as follows (for details on general optimal recovery problems see [8], [9], [16]). Suppose that $f \in X$ is approximately known on $D_1, \dots, D_n$. It is required to find an optimal method of recovery of f on D_0.

This means that we are given $y_1, \dots, y_n$ defined on $D_1, \dots, D_n$ such that

$$\|f_j - y_j\|_{L^2(\mu_j)} \le \delta_j, \quad j = 1, \dots, n,$$

where f_j is the restriction of f to D_j and $\delta_j \ge 0$, $j = 1, \dots, n$ are accuracy levels. In particular, $\delta_j = 0$ means that f is known precisely on D_j.

A recovery algorithm (method, procedure, etc.) is an operator

$$A\colon L^2(\mu_1) \times \ldots \times L^2(\mu_n) \to L^2(\mu_0).$$

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We consider $A(y)$, $y = (y_1, \dots, y_n)$, to be the recovered value of f on D_0. At this point we impose no conditions on A. In particular, we require A to be neither continuous, nor linear.

Given a recovery method A its accuracy is characterized by the maximal possible error

$$\begin{aligned} e(X, \mathcal{D}, \mu, \delta, A) = \sup\{ &\|f_0 - A(y)\|_{L^2(\mu_0)} : f \in X, \\ &y \in L^2(\mu_1) \times \ldots \times L^2(\mu_n),\ \|f_j - y_j\|_{L^2(\mu_j)} \le \delta_j,\ j = 1, \dots, n\}. \end{aligned}$$

We futher introduce the optimal recovery error as

$$E(X, \mathcal{D}, \mu, \delta) = \inf_{A\colon L^2(\mu_1)\times\ldots\times L^2(\mu_n)\to L^2(\mu_0)} e(X, \mathcal{D}, \mu, \delta, A). \tag{1}$$

A method $\widehat{A}$ such that

$$E(X, \mathcal{D}, \mu, \delta) = e(X, \mathcal{D}, \mu, \delta, \widehat{A})$$

is called an *optimal recovery method.*

The problem of finding an optimal recovery method (and sometimes an extremal function at which the optimal recovery error is attained) is usually referred to as *optimal recovery problem.*

As we will show below this problem is closely related to the following extremal problem which we call the *dual problem.* Find

$$\sup\left\{ \|f_0\|^2_{L^2(\mu_0)} : f \in X,\ \|f_j\|^2_{L^2(\mu_j)} \le \delta_j^2,\ j = 1, \dots, n \right\}. \tag{2}$$

In the special case when D is the unit disk, $D = \mathbb{D}$, $n = 2$, measures μ_0, μ_1, μ_2 are the normalized Lebesgue measures on the circles $\{|z| = \rho\}$, $\{|z| = r_1\}$ and $\{|z| = r_2\}$ respectively ($r_1 < \rho < r_2$), and X is the Hardy space H^2, problem 2 is reminiscent of the Hadamard three circle theorem (cf. [15, Chapter 14]) which states that for an analytic function f in the unit disk

$$M(\rho) \le M^{\frac{\log r_2/\rho}{\log r_2/r_1}}(r_1) M^{\frac{\log \rho/r_1}{\log r_2/r_1}}(r_2),$$

where

$$M(r) = \max\{|f(z)| : |z| = r\}.$$

This result gives an estimate for the value of the following extremal problem. Find

$$\max\{\|f_\rho\|_{H^\infty}\ :\ \|f_{r_1}\|_{H^\infty} \le \delta_1,\ \|f_{r_2}\|_{H^\infty} \le \delta_2\},$$

where $f_\tau(z) = f(\tau z)$.

In section 3 of this paper we consider a similar problem in the Hardy space in the unit ball of $\mathbb{C}^n$. We call problems of this form Hadamard type problems.

Another case is when D is the unit disk $\mathbb{D}$, μ_0 and μ_1 are point masses and μ_2 is the normalized Lebesgue measure on the unit circle. Here problem (2) turns into

$$\max\{|f(a_0)| : |f(a_1)| \le \delta_1,\ \|f\|_{H^2} \le \delta_2\}, \tag{3}$$

where $a_0, a_1 \in \mathbb{D}$. A more general problem (with H^p-norm constraint) was considered in [13]. Results of this type might be viewed as generalizations of the classical Schwarz Lemma. In this paper we investigate another generalization of Schwarz Lemma, which is obtained from (3) by the replacement of the point mass at a_0 with the normalized Lebesgue measure on a circle centered at a_0. As we will see this change makes the problem much harder. In particular, it is a rare occasion when the extremal function is rational.

The structure of this paper is as follows. In section 2 we prove two general results, Theorem 1 and Theorem 2. The first of them gives a necessary condition of extremum in the dual problem in terms of inclusion in certain annihilators. The second expresses the value of the dual extremal problem in terms of it's spectrum. These results provide the main tool for our investigation of two extremal problems one of which is of Hadamard type and the other is a generalization of the Schwarz Lemma. These problems are considered in section 3. Here we describe spectra and extremal spectral points for both problems. In section 4 we prove another general theorem, Theorem 6, which gives a way of constructing an optimal recovery method under a certain condition. We show that in our cases this condition is met and use Theorem 6 for the construction of optimal recovery methods in corresponding optimal recovery problems. Finally, section 5 contains some open problems.

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2. Euler's Equation

The main goal of this section is to give a necessary condition in problem (2) in terms of certain annihilators. Let $K(z, w)$ be the reproducing kernel of X. We may consider measures $\mu_0, \mu_1, \ldots, \mu_n$ as defined on the whole domain D by the trivial extension outside of $D_0, D_1, \ldots, D_n$ respectively. Write

$$\widetilde{\mu} = -\mu_0 + \sum_{j=1}^{n} \lambda_j \mu_j.$$

Then $\widetilde{\mu}$ is a regular measure on D and every function from X is square-integrable with respect to $\widetilde{\mu}$. For $w \in D$ we introduce

$$d\widetilde{\mu}_w(z) = K(z,w)d\widetilde{\mu}(z).$$

Then obviously every function from X is $\widetilde{\mu}_w$-integrable. The measures $\widetilde{\mu}$ and $\widetilde{\mu}_w$ depend on $\lambda = (\lambda_1, \dots, \lambda_n)$. We explicitly indicate this dependence for the regular part of $\widetilde{\mu}_w$ and write

$$\tau_w^\lambda(z) = \int_D K(z,\tau)\, d\widetilde{\mu}_w(\tau).$$

Recall that given a convex function g on a convex subset A of a Banach space X, the subdifferential of g at a point $t \in A$, $\partial g(t)$, consists of all continuous linear functional l on X such that for every $x \in A$

$$\langle x - t, l \rangle \leq g(x) - g(t).$$

It is well-known that if A is open and g is continuous at t, then $\partial g(t) \neq \emptyset$.

We will need the following result.

Lemma 1. *Let X be a Banach space and g_1, g_2 be continuous convex positive functions in a convex neighborhood of $\zeta \in X$ and*

$$h = \frac{g_1}{g_2}.$$

If ζ is a point of local maximum of h, then $\partial g_1(\zeta) \subset h(\zeta)\partial g_2(\zeta)$; if ζ is a point of local minimum of h , then $h(\zeta)\partial g_2(\zeta) \subset \partial g_1(\zeta)$.

Proof. Let ζ be the point of local maximum. For every z in a neighborhood of ζ, $g_1(z) \leq h(\zeta) g_2(z)$. If $x^* \in \partial g_1(\zeta)$, then

$$\langle z - \zeta, x^* \rangle \leq g_1(z) - g_1(\zeta) \leq h(\zeta)(g_2(z) - g_2(\zeta)),$$

which means that $x^* \in h(\zeta)\partial g_2(\zeta)$. The other statement is proved in a similar way. □

We apply Lemma 1 to the following special case where

$$g_2 = \max\{\varphi_1, \dots, \varphi_n\},$$

and $\varphi_1, \dots, \varphi_n$ are positive and convex functions in a convex neighborhood of $\zeta \in X$, which are continuous at ζ. In this case the following theorem of Dubovickii and Miljutin [3], (also see [6], English translation in [1], and [4]) expresses the subdifferential of g_2 in terms of subdifferentials of φ_j, $j = 1, ..., n$.

Theorem A. *Let X be a Banach space, $\varphi_j : X \to \mathbb{R} \cup \{\pm\infty\}$, $j = 1, ..., n$ be convex functions on X continuous at ζ,*

$$F(x) = \max\{\varphi_1(x), ..., \varphi_n(x)\},$$

and

$$F(\zeta) = \varphi_{j_1}(\zeta) = \varphi_{j_2}(\zeta) = ... = \varphi_{j_k}(\zeta),$$
$$F(\zeta) > \varphi_l(\zeta), \text{ if } l \neq j_1, ..., j_k.$$

Then

$$\partial F(\zeta) = \operatorname{co} \bigcup_{m=1}^{k} \partial \varphi_{j_m}(\zeta).$$

Theorem A implies that $\partial g_2(\zeta)$ is the convex combination of subdifferentials of those φ_j which coincide with g_2 at ζ. If in addition all functions $g_1, \varphi_1, \ldots, \varphi_n$ are Frechet differentiable at ζ, then their subdifferentials at ζ consist of corresponding Frechet derivatives and we obtain the following

Corollary 1. *Let $g, \varphi_1, \ldots, \varphi_n$ be positive, convex in a convex neighborhood of ζ, and Frechet differentiable at ζ. If ζ is a point of extremum of*

$$h = \frac{g}{\max\{\varphi_1, \ldots, \varphi_n\}},$$

then there are $0 \leq \lambda_1, \ldots, \lambda_n \leq 1$ such that

1. $\displaystyle\sum_{j=1}^{n} \lambda_j = 1.$

2. $\lambda_j \left(\varphi_j(\xi) - \max\{\varphi_1, \ldots, \varphi_n\}\right) = 0.$

3. $\displaystyle g'(\zeta) = \sum_{j=1}^{n} \lambda_j h(\zeta) \varphi_j'(\zeta).$

Now we are ready to prove our result about annihilators.

Theorem 1. *If $\widehat{f} \in X$ is a solution of problem (2), then there exists a non-negative vector $\widehat{\lambda} = (\widehat{\lambda}_1, \ldots, \widehat{\lambda}_n)$ such that*

$$\widehat{f} \in (\operatorname{span}\{\tau_w^{\widehat{\lambda}},\ w \in D\})^{\perp}.$$

and

$$\widehat{\lambda}_j(\|f\|_{L_2(\mu_j)} - \delta_j) = 0, \quad j = 1, \ldots, n.$$

Proof. For $x \in X$ write

$$g(x) = \|\widehat{f} + x\|^2_{L_2(\mu_0)}, \quad \varphi_j(x) = \frac{\|\widehat{f} + x\|^2_{L_2(\mu_j)}}{\delta_j^2}, \quad j = 1, \dots, n.$$

Remark that the function

$$\psi_x(t) = \frac{\widehat{f}(t) + x(t)}{\max\left\{\sqrt{\varphi_1(x)}, \dots, \sqrt{\varphi_n(x)}\right\}}$$

is admissible in problem (2). This implies that the function

$$h(x) = \frac{g(x)}{\max\{\varphi_1(x), \dots, \varphi_n(x)\}}$$

attains its maximum at $x = 0$. Since all the functions $g, \varphi_1, \dots, \varphi_n$ are obviously Frechet differentiable at $x = 0$ (since $\widehat{f}$ is clearly non-trivial), we are in the conditions of Corollary 1, and, therefore, there are $\lambda_1, \dots, \lambda_n$ satisfying statements 1 and 2 of Corollary 1. Since clearly

$$\max_{1 \le j \le n} \left\{ \frac{\|\widehat{f}\|^2_{L_2(\mu_j)}}{\delta_j^2} \right\} = 1,$$

we have $h(0) = \|\widehat{f}\|^2_{L_2(\mu_0)}$. Write

$$\widehat{\lambda}_j = \frac{\|\widehat{f}\|^2_{L_2(\mu_0)}}{\delta_j^2} \lambda_j, \quad j = 1, \dots, n. \tag{4}$$

Then it follows from Corollary 1 that for every $u \in X$

$$-\int_{D_0} u\overline{\widehat{f}}\, d\mu_0 + \sum_{j=1}^{n} \widehat{\lambda}_j \int_{D_j} u\overline{\widehat{f}}\, d\mu_j = \int_D u\overline{\widehat{f}}\, d\widetilde{\mu} = 0. \tag{5}$$

Using Fubini's theorem we obtain

$$\begin{aligned} 0 &= \int_D \overline{\widehat{f}}(z) \left(\int_D u(w) K(z,w)\, d\nu(w) \right) d\widetilde{\mu}(z) \\ &= \int_D u(w) \int_D \overline{\widehat{f}(z) K(w,z)}\, d\widetilde{\mu}(z)\, d\nu(w). \end{aligned}$$

Since

$$\int_D \widehat{f}(z) K(w,z)\, d\widetilde{\mu}(z) \in X,$$

the last equality implies

$$\int_D \widehat{f}(z) K(w,z)\, d\widetilde{\mu}(z) = 0.$$

Finally, for every $w \in D$ we have

$$(6) \quad 0 = \int_D \widehat{f}(z) K(w,z)\, d\widetilde{\mu}(z)$$
$$= \int_D \widehat{f}(\tau) \int_D K(z,\tau) K(w,z)\, d\widetilde{\mu}(z)\, d\nu(\tau) = \langle \widehat{f}, \tau_w^{\widehat{\lambda}} \rangle.$$

□

Note that it follows from (4) that

$$\|\widehat{f}\|^2_{L_2(\mu_0)} = \sum_{j=1}^n \widehat{\lambda}_j \delta_j^2. \tag{7}$$

In reality this equality holds on a much wider set of functions which we call spectral functions of problem (2). They are defined as follows. We say that a non-negative vector $\lambda = (\lambda_1, \ldots, \lambda_n)$ belongs to the *spectrum* of problem (2), if there exists an admissible for problem (2) function $f \in X$ such that

1. $\lambda_j(\|f\|_{L_2(\mu_j)} - \delta_j) = 0$.
2. $f \in (\operatorname{span}\{\tau_w^\lambda,\ w \in D\})^\perp$.

In this case we call f a *spectral function.*

It is very easy to see that if $\lambda = (\lambda_1, \ldots, \lambda_n)$ is in the spectrum of problem (2) and f is a corresponding spectral function, then equation (5) holds, namely

$$-\int_{D_0} u\overline{f}\, d\mu_0 + \sum_{j=1}^n \lambda_j \int_{D_j} u\overline{f}\, d\mu_j = 0.$$

Now the substitution $u = f$ shows that (7) holds for any spectral function, that is

$$\|f\|^2_{L_2(\mu_0)} = \sum_{j=1}^n \lambda_j \delta_j^2.$$

Thus, we obtain the following result

Theorem 2. *Let* Λ *be the spectrum of problem* (2)*. Then*

$$\sup_{\substack{f \in X \\ \|f_j\|_{L^2(\mu_j)} \le \delta_j,\ j=1,\ldots,n}} \|f\|^2_{L^2(\mu_0)} = \sup_{\lambda \in \Lambda} \sum_{j=1}^n \lambda_j \delta_j^2. \tag{8}$$

We call a spectral point $(\widehat{\lambda}_1, \ldots, \widehat{\lambda}_n)$ *extremal*, if the maximum of the right-hand side of (8) is attained at $(\widehat{\lambda}_1, \ldots, \widehat{\lambda}_n)$.

3. Extremal Problems

3.1. **Hadamard Type Problem in the Unit Ball of $\mathbb{C}^n$.** Let $\mathbf{B}_n$ stand for the unit ball in $\mathbb{C}^n$,

$$\mathbf{B}_n = \left\{ z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z|^2 = \sum_{j=1}^{n} |z_j|^2 < 1 \right\}.$$

Recall that the Hardy space $H^p(\mathbf{B}_n)$ consists of all functions f such that

$$\begin{aligned} \|f\|^p_{H^p(\mathbf{B}_n)} &= \sup_{0<r<1} \int_{|z|=1} |f(rz)|^p \, d\sigma(z) < \infty, \quad 1 \le p < \infty, \\ \|f\|_{H^\infty(\mathbf{B}_n)} &= \sup_{z \in \mathbf{B}_n} |f(z)|, \end{aligned}$$

where $d\sigma(z)$ is the positive normalized rotationally invariant measure on the unit sphere

$$S = \{ z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z| = 1 \}.$$

It is well known that $H^p(\mathbf{B}_n)$-functions have radial limits σ-almost everywhere on the unit sphere S (see [14], sect 1.4.9), which are usually denoted by the same letter f and

$$\|f\|^p_{H^p(\mathbf{B}_n)} = \int_{|z|=1} |f(z)|^p \, d\sigma(z).$$

It is also well known that the reproducing kernel for the Hardy space is

$$K(z, w) = \frac{1}{(1 - \langle z, w \rangle)^n}$$

(see [14], sect 7.1.4), so that for every $f \in H^p(\mathbf{B}_n)$ and $w \in \mathbf{B}_n$

$$f(w) = \int_{|z|=1} f(z) \overline{K(z, w)} \, d\sigma(z).$$

Let $0 < r_1 < \rho < r_2 < 1$, $f_r(z) = f(rz)$. Consider the following Hadamard type extremal problem. Find

$$\sup \left\{ \|f_\rho\|_{H^2(\mathbf{B}_n)} : f \in H^2(\mathbf{B}_n), \ \|f_{r_j}\|_{H^2(\mathbf{B}_n)} \le \delta_j, \ j = 1, 2 \right\}. \tag{9}$$

Theorem 3. *Let $0 < r_1 < \rho < r_2 < 1$ and $\delta_1, \delta_2 > 0$. Then*

1. *If $s \in \mathbb{Z}_+$ is such that*

$$\left(\frac{r_1}{r_2} \right)^{s+1} < \frac{\delta_1}{\delta_2} < \left(\frac{r_1}{r_2} \right)^{s},$$

then the unique extremal spectral point of (9) *is*

$$\left(\frac{r_2^2-\rho^2}{r_2^2-r_1^2}\left(\frac{\rho}{r_1}\right)^{2s}, \frac{\rho^2-r_1^2}{r_2^2-r_1^2}\left(\frac{\rho}{r_2}\right)^{2s}\right).$$

2. *If* $\delta_1 > \delta_2$*, then the unique extremal spectral point of* (9) *is* $(0,1)$*.*
3. *If there is* $s \in \mathbb{Z}_+$ *such that*

$$\frac{\delta_1}{\delta_2} = \left(\frac{r_1}{r_2}\right)^s,$$

then the set of extremal spectral point of (9) *is* $(\widehat{\lambda}_1, \widehat{\lambda}_2)$*, where* $\widehat{\lambda}_1, \widehat{\lambda}_2 \geq 0$ *and*

$$\widehat{\lambda}_1 r_1^{2s} + \widehat{\lambda}_2 r_2^{2s} = \rho^{2s}. \tag{10}$$

Proof. Let σ_ρ, σ_{r_1} and σ_{r_2} denote normalized Lebesgue surface area measures on spheres of radii ρ, r_1 andr_2 respectively. Below we consider them as measures on $\mathbf{B}_n$. We have

$$\widetilde{\mu} = -\sigma_\rho + \lambda_1\sigma_{r_1} + \lambda_2\sigma_{r_2}, \quad d\widetilde{\mu}_w(z) = K(z,w)\, d\widetilde{\mu}(z).$$

Thus,

$$\begin{aligned}\tau_w^\lambda &= \int_{|z|\leq 1} K(z,\tau)\, d\widetilde{\mu}_w(\tau) \\ &= \int_{|z|\leq 1} K(z,\tau)K(\tau,w)(-d\sigma_\rho(\tau) + \lambda_1 d\sigma_{r_1}(\tau) + \lambda_2 d\sigma_{r_2}(\tau)).\end{aligned}$$

Making the substitution $\tau = r\eta$, we obtain

$$\begin{aligned}\int_{|z|\leq 1} K(z,\tau)K(\tau,w)\, d\sigma_r(\tau) &= \int_{|z|=1} K(z,r\eta)K(r\eta,w)\, d\sigma(\eta) \\ &= \int_{|z|=1} K(rz,\eta)K(\eta,rw)\, d\sigma(\eta) = K(rz,rw).\end{aligned}$$

Hence

$$\tau_w^\lambda = -\frac{1}{(1-\rho^2\langle z,w\rangle)^n} + \frac{\lambda_1}{(1-r_1^2\langle z,w\rangle)^n} + \frac{\lambda_2}{(1-r_2^2\langle z,w\rangle)^n}.$$

The condition $f \perp \tau_w^\lambda$ for all w means that for all w

$$\int_{|z|=1}\left(-\frac{1}{(1-\rho^2\langle z,w\rangle)^n} + \frac{\lambda_1}{(1-r_1^2\langle z,w\rangle)^n} + \frac{\lambda_2}{(1-r_2^2\langle z,w\rangle)^n}\right) f(z)\, d\sigma(z) = 0.$$

Consequently, for all w

$$-f(\rho^2 w) + \lambda_1 f(r_1^2 w) + \lambda_2 f(r_2^2 w) = 0. \tag{11}$$

If

$$f(z) = \sum_{j=0}^{\infty} \sum_{|\alpha|=j} c_\alpha z^\alpha,$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, and $z^\alpha = z_1^{\alpha_1} \dots z_n^{\alpha_n}$, then (11) means that

$$\sum_{j=0}^{\infty} (-\rho^{2j} + \lambda_1 r_1^{2j} + \lambda_2 r_2^{2j}) \sum_{|\alpha|=j} c_\alpha w^\alpha = 0$$

for all w. It can be easily verified that there are no more than two values of j for which

$$-\rho^{2j} + \lambda_1 r_1^{2j} + \lambda_2 r_2^{2j} = 0.$$

Assume that $m > k$ and

$$\begin{aligned} -\rho^{2k} + \lambda_1 r_1^{2k} + \lambda_2 r_2^{2k} &= 0, \\ -\rho^{2m} + \lambda_1 r_1^{2m} + \lambda_2 r_2^{2m} &= 0. \end{aligned}$$

Then

$$\begin{aligned} \lambda_1 &= \frac{r_2^{2(m-k)} - \rho^{2(m-k)}}{r_2^{2(m-k)} - r_1^{2(m-k)}} \left(\frac{\rho}{r_1}\right)^{2k}, \\ \lambda_2 &= \frac{\rho^{2(m-k)} - r_1^{2(m-k)}}{r_2^{2(m-k)} - r_1^{2(m-k)}} \left(\frac{\rho}{r_2}\right)^{2k}, \end{aligned} \tag{12}$$

and functions satisfying (11) have the following form

$$f(z) = \sum_{|\alpha|=k} c_\alpha z^\alpha + \sum_{|\alpha|=m} c_\alpha z^\alpha.$$

Since

$$\|f(r_j z)\|_{H^2(\mathbf{B}_n)} = \delta_j, \ j = 1, 2, \tag{13}$$

and monomials z^α form an orthogonal system in $H^2(\mathbf{B}_n)$ with

$$\|z^\alpha\|^2_{H^2(\mathbf{B}_n)} = \frac{n!\alpha!}{(n + |\alpha| - 1)!},$$

(see [14], sect. 1.4.9) we have

$$r_j^{2k} d_k + r_j^{2m} d_m = \delta_j^2, \quad j = 1, 2, \tag{14}$$

where

$$d_s = \frac{n!}{(n+s-1)!} \sum_{|\alpha|=s} |c_\alpha|^2 \alpha!, \quad s = k, m. \tag{15}$$

Solving (14) for d_k, d_m we obtain

$$d_k = \frac{\delta_1^2 r_2^{2m} - \delta_2^2 r_1^{2m}}{r_1^{2k} r_2^{2k} \big(r_2^{2(m-k)} - r_1^{2(m-k)}\big)},$$

$$d_m = \frac{\delta_2^2 r_1^{2k} - \delta_1^2 r_2^{2k}}{r_1^{2k} r_2^{2k} \big(r_2^{2(m-k)} - r_1^{2(m-k)}\big)}.$$

Since $d_k, d_m \geq 0$ we have

$$\left(\frac{r_1}{r_2}\right)^m \leq \frac{\delta_1}{\delta_2} \leq \left(\frac{r_1}{r_2}\right)^k.$$

Assume that for some $s = 0, 1, \ldots,$

$$\left(\frac{r_1}{r_2}\right)^{s+1} < \frac{\delta_1}{\delta_2} < \left(\frac{r_1}{r_2}\right)^s. \tag{16}$$

It follows from Theorem 2 that in order to find extremal spectral points we are to find

$$\sup_{\substack{k,m \in \mathbb{Z}_+ \\ k \leq s,\ m \geq s+1}} (\lambda_1 \delta_1^2 + \lambda_2 \delta_2^2),$$

where λ_1, λ_2 are defined by (12). We have

$$\lambda_1 \delta_1^2 + \lambda_2 \delta_2^2 = \lambda_1 \delta_2^2 \left(\left(\frac{\delta_1}{\delta_2}\right)^2 - \left(\frac{r_1}{r_2}\right)^{2k} \right) + \frac{\rho^{2k} \delta_2^2}{r_2^{2k}}.$$

Fix $k \leq s$. Then we can write

$$\lambda_1 = \omega_\alpha(t) \left(\frac{\rho}{r_1}\right)^{2k},$$

where

$$\omega_\alpha(t) = \frac{1 - t^\alpha}{1 - t}, \quad t = \left(\frac{r_1}{r_2}\right)^{2(m-k)}, \quad \alpha = \frac{\log \rho / r_2}{\log r_1 / r_2}.$$

Observe that $0 < \alpha < 1$. It can be easily shown that $\omega_\alpha(t)$ is a decreasing function for $0 < t < 1$. Thus, λ_1 increases as $m \to \infty$. Since

$$\left(\frac{\delta_1}{\delta_2}\right)^2 - \left(\frac{r_1}{r_2}\right)^{2k} < 0,$$

for a fixed $k \leq s$ the maximum of $\lambda_1 \delta_1^2 + \lambda_2 \delta_2^2$ is attained at $m = s + 1$.

Now fix $m = s+1$. Using the representations

$$\lambda_2 = \omega_\beta(t)\left(\frac{\rho}{r_2}\right)^{2m}, \quad \beta = \frac{\log \rho/r_1}{\log r_2/r_1},$$

$$\lambda_1\delta_1^2 + \lambda_2\delta_2^2 = \lambda_2\delta_1^2\left(\left(\frac{\delta_2}{\delta_1}\right)^2 - \left(\frac{r_2}{r_1}\right)^{2m}\right) + \frac{\rho^{2m}\delta_1^2}{r_1^{2m}},$$

we obtain that the maximum of $\lambda_1\delta_1^2 + \lambda_2\delta_2^2$ is attained at $k = s$. Thus,

$$\widehat{\lambda}_1 = \frac{r_2^2 - \rho^2}{r_2^2 - r_1^2}\left(\frac{\rho}{r_1}\right)^{2s},$$

$$\widehat{\lambda}_2 = \frac{\rho^2 - r_1^2}{r_2^2 - r_1^2}\left(\frac{\rho}{r_2}\right)^{2s},$$

Suppose that for some $s = 1, 2, \dots$

$$\frac{\delta_1}{\delta_2} = \left(\frac{r_1}{r_2}\right)^s.$$

Then

$$\sup_{\substack{k,m\in\mathbb{Z}_+\\ k\le s,\ m\ge s+1}} (\lambda_1\delta_1^2 + \lambda_2\delta_2^2) = \sup_{\substack{k,m\in\mathbb{Z}_+\\ k\le s-1,\ m\ge s}} (\lambda_1\delta_1^2 + \lambda_2\delta_2^2) = \delta_2^2\left(\frac{\rho}{r_2}\right)^{2s},$$

and the coordinates of extremal points $(\widehat{\lambda}_1, \widehat{\lambda}_2)$ satisfy equality (10). Thus, this situation is included in the case 3.

If $\delta_1 = \delta_2$ $(s = 0)$, then $k = 0$ and

$$\sup_{\substack{m\in\mathbb{Z}\\ m\ge 1}} (\lambda_1\delta_1^2 + \lambda_2\delta_2^2) = \delta_1^2(\widehat{\lambda}_1 + \widehat{\lambda}_2) = \delta_1^2,$$

since

$$\widehat{\lambda}_1 = \frac{r_2^2 - \rho^2}{r_2^2 - r_1^2}, \quad \widehat{\lambda}_2 = \frac{\rho^2 - r_1^2}{r_2^2 - r_1^2}.$$

That is, again this situation is described in the case 3.

Finally, suppose that there is only one s such that

$$-\rho^{2s} + \lambda_1 r_1^{2s} + \lambda_2 r_2^{2s} = 0.$$

Then any function satisfying (11) has the following form

$$f(z) = \sum_{|\alpha|=s} c_\alpha z^\alpha.$$

If $\lambda_1, \lambda_2 > 0$, then it follows from (13) that

$$\frac{\delta_1}{\delta_2} = \left(\frac{r_1}{r_2}\right)^s.$$

Thus, for all $\widehat{\lambda}_1, \widehat{\lambda}_2 > 0$ satisfying (10)

$$\widehat{\lambda}_1\delta_1^2 + \widehat{\lambda}_2\delta_2^2 = \delta_2^2\left(\widehat{\lambda}_1\frac{\delta_1^2}{\delta_2^2} + \widehat{\lambda}_2\right) = \delta_2^2\left(\frac{\rho}{r_2}\right)^{2s}.$$

The case when $\lambda_1 = 0$ or $\lambda_2 = 0$ may be considered in a similar way. □

3.2. **Generalized Schwarz lemma in the Hardy space.** Recall that the classical Schwarz lemma states that an analytic function f which takes the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ into itself and vanishes at the origin, satisfies the inequality

$$|f(z)| \leq |z|$$

for all z in the unit disk. Of course, this result is strongly related to the following extremal problem. Given $a \in \mathbb{D}$ find

$$\sup_{\substack{f\in H^\infty\\ f(0)=0}} |f(a)|.$$

There are several papers where similar results were considered for Hardy and Bergman spaces in connection with optimal recovery in both one and several dimensional cases (see, for example, [11]–[13]).

Here we consider the following problem. Let $a \in \mathbb{D}$ and Γ be a circle inside of the unit disk, μ be the normalized Lebesgue measure on Γ, and $\delta > 0$. Find

$$\sup\left\{ \int_\Gamma |f|^2\, d\mu : f \in H^2,\ \|f\|_{H^2} \leq 1,\ |f(a)| \leq \delta \right\}. \tag{17}$$

The special case of this problem when Γ degenerates to a point was considered in [13].

To simplify the notation we will consider the case when the circle Γ passes through the origin and its center lies on the real axis, so that

$$\Gamma = \{\, z \in \mathbb{C} : |z - \rho| = \rho \,\},$$

$0 < \rho < 1/2$. In general case the argument goes along similar lines but computations are longer.

To solve problem (17) we once again use Theorem 1. In our case the measure μ_0 is the normalized Lebesgue measure on Γ, μ_1 is the unit point mass at a, and μ_2 is the normalized Lebesgue measure on the unit circle $\mathbb{T}$. Thus,

$$\tau_w^\lambda = -\frac{1}{2\pi}\int_\Gamma \frac{1}{1-z\bar\tau}\cdot\frac{1}{1-\tau\bar w}\frac{|d\tau|}{|\tau-\rho|} + \lambda_1\frac{1}{1-z\bar a}\cdot\frac{1}{1-a\bar w}$$
$$+\frac{\lambda_2}{2\pi}\int_{|\tau|=1}\frac{1}{1-z\bar\tau}\cdot\frac{1}{1-\tau\bar w}|d\tau|$$
$$= -\frac{1}{1-z\rho-\rho\bar w} + \frac{\lambda_1}{(1-z\bar a)(1-a\bar w)} + \frac{\lambda_2}{1-z\bar w}.$$

The existence of an extremal function in problem (17) easily follows from the standard compactness argument. By Theorem 1 every extremal function satisfies the following equation

$$\frac{1}{1-\rho w}f\left(\frac{\rho}{1-\rho w}\right) = \lambda_1\frac{f(a)}{1-\bar a w} + \lambda_2 f(w) \tag{18}$$

for some $\lambda_1, \lambda_2 \ge 0$ and all $w \in \mathbb{D}$. Our next step is to describe the spectrum and spectral functions of problem (17). Spectral functions are H^2 functions of norm not exceeding 1 satisfying the condition $|f(a)| \le \delta$ and equation (18).

Let

$$b = \frac{1-\sqrt{1-4\rho^2}}{2\rho}.$$

It is easily verified that b is the Denjoy-Wolff point (see [2]) of the following self-mapping of $\mathbb{D}$

$$z \mapsto \frac{\rho}{1-\rho z}.$$

It is also easy to see that the disk bounded by the circle Γ is a hyperbolic neighborhood of b. Consider the following functions

$$\varphi_j(z) = \frac{\sqrt{1-b^2}}{1-bz}\left(\frac{b-z}{1-bz}\right)^j, \quad j = 0, 1, \dots . \tag{19}$$

These functions form an orthonormal system in H^2, and, since any H^2-function which is orthogonal to all φ_j must vanish at b together with all its derivatives, they form an orthonormal basis of H^2. Moreover, they are eigenfunctions of the operator

$$Tf(z) = \frac{1}{1-\rho z}f\left(\frac{\rho}{1-\rho z}\right). \tag{20}$$

Indeed, using the fact that

$$\frac{\rho}{1-\rho b} = b, \tag{21}$$

we have

$$T\varphi_j(z) = \frac{1}{1-\rho z} \cdot \frac{\sqrt{1-b^2}}{1-b\dfrac{\rho}{1-\rho z}} \left(\frac{b - \dfrac{\rho}{1-\rho z}}{1-b\dfrac{\rho}{1-\rho z}} \right)^j$$

$$= \frac{1}{1-b\rho} \frac{\sqrt{1-b^2}}{1-bz} \left(\frac{(b-\rho)-b\rho z}{(1-b\rho)-b\rho z} \right)^j = \alpha_j \varphi_j(z),$$

where

$$\alpha_j = \frac{b^{2j}}{1-\rho b}. \tag{22}$$

Note in passing (though we will not use it explicitly) that the above argument shows that the operator T is self-adjoint.

The next two theorems give a description of the spectrum of the problem (17).

Theorem 4. *Let $a \neq b$. 1. If*

$$\left| a - \frac{\rho}{1-\rho^2} \right| \geq \frac{\rho^2}{1-\rho^2},$$

or

$$\delta > \frac{\sqrt{|a|^2\rho^2 - |\rho - a|^2}}{a\rho + \bar{a}\rho - |a|^2},$$

then the spectrum of problem (17) *consists of two parts $\Lambda = \Lambda_1 \cup \Lambda_2$, where*

$$\Lambda_1 = \{ (0, \alpha_j) : |\varphi_j(a)| \leq \delta \},$$
$$\Lambda_2 = \left\{ (\lambda_1, \lambda_2) : \lambda_1, \lambda_2 > 0, \quad F(\lambda_2) = \delta^{-2}, \quad \lambda_1 = h(\lambda_2) \right\},$$

where

$$F(\lambda) = \sum_{j=0}^{\infty} \frac{|\varphi_j(a)|^2}{(\alpha_j - \lambda)^2} h^2(\lambda), \quad h(\lambda) = \left(\sum_{j=0}^{\infty} \frac{|\varphi_j(a)|^2}{\alpha_j - \lambda} \right)^{-1}.$$

2. *If*

$$\left| a - \frac{\rho}{1-\rho^2} \right| < \frac{\rho^2}{1-\rho^2}, \tag{23}$$

and

$$\delta \leq \frac{\sqrt{|a|^2\rho^2 - |\rho - a|^2}}{a\rho + \bar{a}\rho - |a|^2}, \tag{24}$$

then the spectrum of problem (17) *includes in addition the point*

$$\Lambda_3 = \left\{ \left(\frac{a\rho + \bar{a}\rho - |a|^2}{\rho^2}, 0 \right) \right\}.$$

Proof. There are the following three possible cases: I. $\lambda_1 = 0$, II. both λ_1 and λ_2 are non-trivial. III. $\lambda_2 = 0$.

I. $\lambda_1 = 0$. In this case the corresponding spectral functions are eigenfunctions of operator (20), defined by (19) for which

$$|\varphi_j(a)| = \frac{\sqrt{1-b^2}}{|1-ab|} \left| \frac{b-a}{1-ab} \right|^j \le \delta.$$

This shows that Λ_1 is a part of the spectrum.

II. Let λ_1 and λ_2 be non-trivial. Write the decomposition of the Cauchy kernel centered at a in the basis $\{\varphi_j\}$

$$\frac{1}{1-\bar{a}w} = \sum_{j=0}^{\infty} \overline{\varphi_j(a)} \varphi_j(w) = \sum_{j=0}^{\infty} \frac{\sqrt{1-b^2}}{1-\overline{a}b} \left(\frac{b-\overline{a}}{1-\overline{a}b} \right)^j \varphi_j(w). \tag{25}$$

Suppose that c_j-s are the Fourier coefficients of f in the basis $\{\varphi_j\}$, that is

$$f = \sum_{j=0}^{\infty} c_j \varphi_j.$$

Equations (18) and (25) imply

$$\alpha_j c_j = \lambda_1 f(a) \overline{\varphi_j(a)} + \lambda_2 c_j. \tag{26}$$

Since $a \neq b$, $\varphi_j(a) \neq 0$. Further, $\lambda_1 \neq 0$ implies $|f(a)| = \delta \neq 0$. Therefore, $\lambda_2 \neq \alpha_j$. Hence,

$$c_j = \lambda_1 \frac{f(a)\overline{\varphi_j(a)}}{\alpha_j - \lambda_2}, \quad f(a) = f(a)\lambda_1 \sum_{j=0}^{\infty} \frac{|\varphi_j(a)|^2}{\alpha_j - \lambda_2},$$

$$\lambda_1 = h(\lambda_2). \tag{27}$$

The condition $\lambda_1 > 0$ implies that λ_2 can not be bigger than α_0. Since $\lambda_2 \neq 0$, $\|f\|_{H^2} = 1$. Thus,

$$F(\lambda_2) = \delta^{-2}. \tag{28}$$

III. $\lambda_2 = 0$. In this case (18) turns into

$$\frac{1}{1-\rho z} f\left(\frac{\rho}{1-\rho z} \right) = \lambda_1 \frac{f(a)}{1-\bar{a}z}.$$

Substituting $w = \dfrac{\rho}{1-\rho z}$ we obtain

$$f(w) = \frac{\lambda_1 f(a)\rho^2}{\bar{a}\rho + (\rho - \bar{a})w}.$$

This function is in H^2 if and only if

$$\left| a - \frac{\rho}{1-\rho^2} \right| < \frac{\rho^2}{1-\rho^2}. \tag{29}$$

If $w = a$, this implies

$$\lambda_1 = \frac{a\rho + \bar{a}\rho - |a|^2}{\rho^2}.$$

It is easy to show that the condition $\| f \|_{H^2} \leq 1$ yields

$$\delta \leq \frac{\sqrt{|a|^2\rho^2 - |\rho - a|^2}}{a\rho + \bar{a}\rho - |a|^2}. \tag{30}$$

If equations (29) and (30) are satisfied, then $\left(\dfrac{a\rho + \bar{a}\rho - |a|^2}{\rho^2}, 0 \right)$ is a spectral point of the problem (17) and

$$f(w) = \frac{(a\rho + \bar{a}\rho - |a|^2)\delta}{\bar{a}\rho + (\rho - \bar{a})w}$$

is the corresponding spectral function. □

Theorem 5. *Let* $a = b$,

$$\Lambda_1 = \{ (0, \alpha_j) : \ j = 1, 2, ... \},$$
$$\Lambda_2 = \left\{ ((1-b^2)(\alpha_0 - \alpha_j), \alpha_j), \ j = 1, 2, \ldots \right\}.$$

Then the spectrum of problem (17) *is* $\Lambda = \Lambda_1 \cup \Lambda_2$, *if* $\delta < \dfrac{1}{\sqrt{1-b^2}}$, *and* $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \{(0, \alpha_0)\}$, *if* $\delta \geq \dfrac{1}{\sqrt{1-b^2}}$.

Proof. In the considered case $\varphi_j(a) = 0$, $j = 1, 2, \ldots$, so the same argument as in the beginning of the proof of the previous theorem shows that Λ_1 in the spectrum and $(0, \alpha_0)$ is in the spectrum if $\delta \geq |\varphi_0(b)| = \dfrac{1}{\sqrt{1-b^2}}$.

Further, in our case (26) turns into

$$(\alpha_0-\lambda_2)c_0=\lambda_1 f(a)\frac{1}{\sqrt{1-b^2}}=\frac{\lambda_1 c_0}{1-b^2},$$
$$(\alpha_j-\lambda_2)c_j=0,\quad j=1,2,\dots .$$

Now the statement about the Λ_2-part of the spectrum is straightforward. □

We will show below that Λ_2 is the most important part of the spectrum. Equations (27) and (28) determine Λ_2. In general, these equations may have infinite number of solutions, but this does not happen if the point a lies outside Γ.

Proposition 1. *If a lies outside Γ, then $F(\lambda)\to\infty$ as $\lambda\to 0$.*

Proof. Observe that if a lies outside Γ, then

$$\left|\frac{b-a}{1-ab}\right|\ge b.$$

Write

$$\gamma=\left|\frac{b-a}{1-ab}\right|, \tag{31}$$

$$\hat F(\lambda)=\frac{\sum_{j=0}^\infty \frac{\gamma^{2j}}{(b^{2j}-\lambda)^2}}{\left(\sum_{j=0}^\infty \frac{\gamma^{2j}}{b^{2j}-\lambda}\right)^2}.$$

Then

$$F(\lambda)=\frac{|1-ab|^2}{1-b^2}\hat F(\lambda(1-b\rho)).$$

Thus, it suffices to prove that $\hat F(\lambda)\overset{\lambda\to 0}{\longrightarrow}\infty$. First, we note that

$$\hat F(b^{2k})=\frac{1}{\gamma^{2k}}\overset{k\to\infty}{\longrightarrow}\infty.$$

Let $b^{2k+2}<\lambda<b^{2k}$. We have

$$\left|\sum_{j=0}^\infty\frac{\gamma^{2j}}{b^{2j}-\lambda}\right|\le\sum_{j=0}^\infty\frac{\gamma^{2j}}{|b^{2j}-\lambda|}.$$

Obviously, for such λ

$$\frac{\gamma^{2k+2}}{\lambda-b^{2k+2}}\ge\frac{\gamma^{2k+2}}{b^{2k+2}}\frac{b^2}{1-b^2},$$

or

$$\frac{\gamma^{2k+2}}{b^{2k+2}} \le \frac{1-b^2}{b^2}\frac{\gamma^{2k+2}}{\lambda - b^{2k+2}}. \tag{32}$$

Now (32) yields

$$\sum_{j=k+2}^{\infty} \frac{\gamma^{2j}}{\lambda - b^{2j}} \le \sum_{j=k+2}^{\infty} \frac{\gamma^{2j}}{\lambda - b^{2k+4}} \le \sum_{j=k+2}^{\infty} \frac{\gamma^{2j}}{b^{2k+2} - b^{2k+4}}$$
$$< \frac{\gamma^{2k+2}}{b^{2k+2}(1-\gamma^2)(1-b^2)} \le \frac{1}{b^2(1-\gamma^2)}\frac{\gamma^{2k+2}}{\lambda - b^{2k+2}}.$$

Further,

$$\sum_{j=0}^{k-1} \frac{\gamma^{2j}}{b^{2j}-\lambda} \le \sum_{j=0}^{k-1} \frac{\gamma^{2j}}{b^{2j}-b^{2k}} \le \frac{1}{1-b^2}\frac{\frac{\gamma^{2k}}{b^{2k}}-1}{\frac{\gamma^2}{b^2}-1} < \frac{b^2}{(1-b^2)(\gamma^2-b^2)}\frac{\gamma^{2k}}{b^{2k}}$$

Also,

$$\frac{\gamma^{2k}}{b^{2k}-\lambda} \ge \frac{1}{1-b^2}\frac{\gamma^{2k}}{b^{2k}}.$$

Therefore,

$$\sum_{j=0}^{k-1} \frac{\gamma^{2j}}{b^{2j}-\lambda} \le \frac{b^2}{\gamma^2-b^2}\frac{\gamma^{2k}}{b^{2k}-\lambda}.$$

Finally, we see that there is a constant M independent of k such that if $b^{2k+2} < \lambda < b^{2k}$, then

$$\left|\sum_{j=0}^{\infty} \frac{\gamma^{2j}}{b^{2j}-\lambda}\right| \le M\left(\frac{\gamma^{2k}}{b^{2k}-\lambda} + \frac{\gamma^{2k+2}}{\lambda - b^{2k+2}}\right), \tag{33}$$

and, therefore,

$$\hat{F}(\lambda) \ge \frac{\frac{\gamma^{2k}}{(b^{2k}-\lambda)^2} + \frac{\gamma^{2k+2}}{(\lambda-b^{2k+2})^2}}{M^2\left(\frac{\gamma^{2k}}{b^{2k}-\lambda} + \frac{\gamma^{2k+2}}{\lambda-b^{2k+2}}\right)^2} \tag{34}$$
$$\ge \frac{\frac{\gamma^{2k}}{(b^{2k}-\lambda)^2} + \frac{\gamma^{2k+2}}{(\lambda-b^{2k+2})^2}}{2M^2\left(\frac{\gamma^{4k}}{(b^{2k}-\lambda)^2} + \frac{\gamma^{4k+4}}{(\lambda-b^{2k+2})^2}\right)}$$
$$\ge \frac{1}{2M^2\gamma^{2k}} \overset{k\to\infty}{\longrightarrow} \infty.$$

□

Corollary 2. *If a lies outside Γ, then there is only a finite number of spectral points with $\lambda_1\lambda_2 \neq 0$.*

Proof. Note that the function

$$g(\lambda) = \sum_{j=0}^{\infty} \frac{|\varphi_j(a)|^2}{\alpha_j - \lambda} \tag{35}$$

is monotone and increases from $-\infty$ to $+\infty$ when $\lambda \in (\alpha_{j+1}, \alpha_j)$. Let ζ_j be the only zero of g on the interval (α_{j+1}, α_j). $F(\lambda)$ is analytic in (ζ_{j+1}, ζ_j) and has poles at the endpoints of this interval. This implies that equation (28) has at most finitely many solutions in each interval (ζ_{j+1}, ζ_j), $j = 0, 1, \dots$. Now the result follows directly from Proposition 1.

□

Now we will use Theorem 2 to describe the extremal points of the spectrum.

Proposition 2. *If $\delta \geq |\varphi_0(a)|$, then $(0, \alpha_0)$ is the extremal point of the spectrum.*

Proof. We claim that φ_0 is the solution of the similar extremal problem without any constraint at a

$$\sup\left\{ \int_\Gamma |f|^2\, d\mu : f \in H^2,\ \|f\|_{H^2} \leq 1, \right\}. \tag{36}$$

Indeed, a standard compactness argument shows that problem (36) has a solution. Theorem 1 implies that Euler's equations of (36) has the form (18) with $\lambda_1 = 0$. Now Theorem 2 implies that α_0 is the maximum value for problem (36) and φ_0 is the function where the maximum is attained.

The condition $\delta \geq |\varphi_0(a)|$ implies that φ_0 is admissible for problem (17), and the result follows. □

Proposition 3. *If $a = b$ and $\delta < 1/\sqrt{1-b^2}$, then the extremal spectral point of (17) is*

$$(\widehat{\lambda}_1, \widehat{\lambda}_2) = ((1-b^2)(\alpha_0 - \alpha_1), \alpha_1).$$

Proof. The proof follows directly from Theorems 2 and 5. □

Let us show that if $\delta < |\varphi_0(a)|$, then Λ_1 does not contain extremal points of the spectrum. We will use the following result.

Proposition 4. *Let $a \neq b$. For every $\delta < |\varphi_0(a)|$ equation* (28) *has exactly one solution on the interval* (ζ_0, α_0).

Proof. It was shown in Corollary 2 that the function

$$g(\lambda) = \sum_{j=0}^{\infty} \frac{|\varphi_j(a)|^2}{\alpha_j - \lambda}$$

monotonically increases when $\lambda \in (\alpha_1, \alpha_0)$, vanishes at ζ_0, and $g(\lambda) \geq 0$, if $\lambda \in (\zeta_0, \alpha_0)$.

Consider the function $F(\lambda)$ from Proposition 1 for $\lambda \in (\zeta_0, \alpha_0)$. Then

$$F'(\lambda) = G(\lambda) h^3(\lambda),$$

where

$$G(\lambda) = \sum_{j=0}^{\infty} \frac{|\varphi_j(a)|^2}{(\alpha_j - \lambda)^3} \sum_{j=0}^{\infty} \frac{|\varphi_j(a)|^2}{\alpha_j - \lambda} - \left(\sum_{j=0}^{\infty} \frac{|\varphi_j(a)|^2}{(\alpha_j - \lambda)^2} \right)^2 .$$

For every $\lambda \in (\xi_0, \alpha_0)$ we have

$$\begin{aligned} G(\lambda) = {} & \frac{|\varphi_0(a)|^2}{(\alpha_0 - \lambda)^3} \sum_{j=1}^{\infty} \frac{|\varphi_j(a)|^2}{\alpha_j - \lambda} + \sum_{j=1}^{\infty} \frac{|\varphi_j(a)|^2}{(\alpha_j - \lambda)^3} \sum_{j=0}^{\infty} \frac{|\varphi_j(a)|^2}{\alpha_j - \lambda} \\ & - 2 \frac{|\varphi_0(a)|^2}{(\alpha_0 - \lambda)^2} \sum_{j=1}^{\infty} \frac{|\varphi_j(a)|^2}{(\alpha_j - \lambda)^2} - \left(\sum_{j=1}^{\infty} \frac{|\varphi_j(a)|^2}{(\alpha_j - \lambda)^2} \right)^2 < 0. \end{aligned}$$

Since, obviously, $F(\alpha_0) = \dfrac{1}{|\varphi_0(a)|^2}$ and $F(\lambda) \to +\infty$ as $\lambda \to \xi_0+$, the result follows. ☐

Proposition 5. *If $\delta < |\varphi_0(a)|$, then Λ_1 does not contain extremal spectral points.*

Proof. First, suppose that $a \neq b$. It suffices to prove that

$$\max_{(\lambda_1, \lambda_2) \in \Lambda} (\lambda_1 \delta^2 + \lambda_2) > \alpha_1. \tag{37}$$

It follows from Proposition 4 that equation (28) has a solution λ_2^* between ζ_0 and α_0. Let $(\lambda_1^*, \lambda_2^*)$ be the corresponding spectral point. Since $\zeta_0 > \alpha_1$, $\lambda_1^* \delta^2 + \lambda_2^* > \alpha_1$.

If $a = b$ the result follows from Proposition 3. ☐

Proposition 6. *Let $a \neq b$. If $\delta \leq |\varphi_1(a)|$, then the extremal spectral point $(\widehat{\lambda}_1, \widehat{\lambda}_2)$ is unique, belongs to Λ_2 and is determined by the condition $\zeta_0 < \widehat{\lambda}_2 < \alpha_0$.*

Proof. It follows from (27) and (28) that

$$\lambda_1\delta^2+\lambda_2=\frac{\delta}{\sqrt{\displaystyle\sum_{j=0}^{\infty}\frac{|\varphi_j(a)|^2}{(\alpha_j-\lambda_2)^2}}}+\lambda_2\le\frac{\delta|\alpha_1-\lambda_2|}{|\varphi_1(a)|}+\lambda_2. \tag{38}$$

For $\lambda_2\le\alpha_1$ the right-hand side of (38) is an increasing linear function of λ_2 which is equal to α_1 at $\lambda_2=\alpha_1$. We see that the function $\lambda_1\delta^2+\lambda_2$ restricted to Λ_2 attains its maximum at a point $(\widehat{\lambda}_1,\widehat{\lambda}_2)$, where $\widehat{\lambda}_2\in(\alpha_1,\alpha_0)$. Moreover, since $\lambda_1\ge 0$, $\widehat{\lambda}_2\in(\zeta_0,\alpha_0)$. The existence and uniqueness of such a point follow from Proposition 4.

Finally let us show that even if the point Λ_3 belongs to the spectrum, the condition $\delta\le|\varphi_1(a)|$ prevents this point from being the point of maximum of the linear form $\lambda_1\delta^2+\lambda_2$.

First we observe that the disk (23) is the hyperbolic neighborhood of b given by

$$\left|\frac{b-a}{1-ab}\right|<b^2. \tag{39}$$

Suppose that a is in the disk (23). We will show that

$$\lambda_1\delta^2<\alpha_1.$$

Then the same arguments as the one which precedes Proposition 4 shows that the point Λ_3 cannot be the point of maximum. Equations (21) and (39) imply

$$\begin{aligned}\lambda_1\delta^2&=\frac{a\rho+\bar{a}\rho-|a|^2}{\rho^2}\delta^2\le\frac{\rho^2-|\rho-a|^2}{\rho^2}|\varphi_1(a)|^2\\&=\frac{\rho^2-|\rho-a|^2}{\rho^2}\frac{1-b^2}{|1-ab|^2}\left|\frac{b-a}{1-ab}\right|^2<\frac{\rho^2-|\rho-a|^2}{\rho^2}\frac{1-b^2}{|1-ab|^2}b^4\end{aligned}$$

We are to show that the right-hand side of the last inequality does not exceed

$$\alpha_1=\frac{b^2}{1-b\rho}=\frac{b^3}{\rho}.$$

That is, we are to prove that

$$\frac{\rho^2-|\rho-a|^2}{\rho}\frac{1-b^2}{|1-ab|^2}b<1.$$

Since

$$|1-ab|\ge 1-|a|b,\quad |\rho-a|\ge\rho-|a|,$$

it suffices to show that

$$\frac{\rho^2 - (\rho - |a|)^2}{\rho} \frac{1 - b^2}{(1 - |a|b)^2} b < 1. \tag{40}$$

Using equation (21), it is easy to verify that the maximum of the function

$$\frac{\rho^2 - (\rho - |a|)^2}{(1 - |a|b)^2}$$

as a function of $|a|$ is attained at $|a| = b$. Now substituting $|a| = b$ into the left-hand side of (40) and once again using the relation $\rho = b(1 + b^2)^{-1}$, we obtain

$$\frac{(2\rho b - b^2)b}{(1 - b^2)\rho} = b^2 < 1,$$

which is exactly what we wanted to show. □

Proposition 7. *Assume that* $|\varphi_1(a)| < \delta < |\varphi_0(a)|$ *and*

$$\gamma = \left| \frac{b - a}{1 - ab} \right| \geq b^{2/3},$$

then the conclusion of Proposition 6 is valid, that is, the extremal spectral point $(\widehat{\lambda}_1, \widehat{\lambda}_2)$ *is unique, belongs to* Λ_2 *and is determined by the condition that* $\zeta_0 < \widehat{\lambda}_2 < \alpha_0$.

Proof. Using the fact that $g(\zeta_0) = 0$, we have

$$\begin{aligned} \frac{|\varphi_0(a)|^2}{\alpha_0 - \zeta_0} = \sum_{j=1}^{\infty} \frac{|\varphi_j(a)|^2}{\zeta_0 - \alpha_0} = \frac{1}{\zeta_0} \sum_{j=1}^{\infty} \frac{|\varphi_0(a)|^2 \gamma^{2j}}{1 - \dfrac{\alpha_0}{\zeta_0}} \\ \geq \frac{|\varphi_0(a)|^2}{\zeta_0} \sum_{j=1}^{\infty} \gamma^{2j} = \frac{|\varphi_0(a)|^2 \gamma^2}{\zeta_0(1 - \gamma^2)}. \end{aligned}$$

Consequently,

$$\frac{1}{\alpha_0 - \zeta_0} \geq \frac{\gamma^2}{\zeta_0(1 - \gamma^2)},$$

which yields $\zeta_0 \geq \alpha_0 \gamma^2$.

On the other hand, since $\gamma \geq b^{2/3}$ it follows from (38) that for all spectral points from Λ_2 such that $\lambda_2 \leq \alpha_1$

$$\lambda_1 \delta^2 + \lambda_2 \leq \frac{\delta \alpha_1}{|\varphi_0(a)|} < \frac{|\varphi_0(a)| \alpha_1}{|\varphi_0(a)|} = \frac{\alpha_0 b^2}{\gamma} \leq \gamma^2 \alpha_0 \leq \zeta_0.$$

It follows from Proposition 4 that there is the spectral point $(\widehat{\lambda}_1, \widehat{\lambda}_2) \in \Lambda_2$ such that $\zeta_0 < \widehat{\lambda}_2 < \alpha_0$. Then

$$\widehat{\lambda}_1\delta^2 + \widehat{\lambda}_2 > \widehat{\lambda}_2 > \zeta_0.$$

□

4. Optimal Recovery Method

In this section we construct optimal recovery methods corresponding to the extremal problems considered in the previous section. We begin with the following general result which will be our main tool (several results of this type may be found in [7], [5], [10]).

Theorem 6. *Assume that there exist $\widehat{\lambda}_j \geq 0$, $j = 1, \dots, n$, such that the value of the extremal problem*

$$\|f_0\|^2_{L^2(\mu_0)} \to \max, \quad \sum_{j=1}^n \widehat{\lambda}_j \|f_j\|^2_{L^2(\mu_j)} \leq \sum_{j=1}^n \widehat{\lambda}_j \delta_j^2, \quad f \in X, \tag{41}$$

is the same as in (2). Moreover, assume that for every $\widetilde{y} = (\widetilde{y}_1, \dots, \widetilde{y}_n) \in Y_1 \times \ldots \times Y_n$, where Y_j are almost everywhere dense in $L^2(\mu_j)$, there exists $f_{\widetilde{y}}$ which is a solution of the extremal problem

$$\sum_{j=1}^n \widehat{\lambda}_j \|f_j - \widetilde{y}_j\|^2_{L^2(\mu_j)} \to \min, \quad f \in X. \tag{42}$$

Moreover, let $\widehat{A}\colon L^2(\mu_1) \times \ldots \times L^2(\mu_n) \to L^2(\mu_0)$ be a linear continuous operator, where the norm in $L^2(\mu_1) \times \ldots \times L^2(\mu_n)$ is defined as

$$\|y\| = \left(\sum_{j=1}^n \|y_j\|^2_{L^2(\mu_j)} \right)^{1/2},$$

such that for all $\widetilde{y} = (\widetilde{y}_1, \dots, \widetilde{y}_n) \in Y_1 \times \ldots \times Y_n$

$$\widehat{A}(\widetilde{y}) = (f_{\widetilde{y}})_0.$$

Then

$$E(X, \mathcal{D}, \mu, \delta) = \sup_{\substack{f \in X \\ \|f_j\|_{L^2(\mu_j)} \leq \delta_j,\ j=1,\dots,n}} \|f_0\|_{L^2(\mu_0)}$$

and the method $\widehat{A}(y)$ is optimal.

Proof. For every method A and for every $f \in X$ such that $\|f_j\|_{L^2(\mu_j)} \leq \delta_j$, $j = 1, \dots, n$, we have

$$2\|f_0\|_{L^2(\mu_0)} \leq \|f_0 - A(0)\|_{L^2(\mu_0)} + \|-f_0 - A(0)\|_{L^2(\mu_0)} \leq 2e(X, \mathcal{D}, \mu, \delta, A).$$

Hence, for every method A

$$e(X,\mathcal{D},\mu,\delta,A) \geq \sup_{\substack{f\in X\\ \|f_j\|_{L^2(\mu_j)}\leq\delta_j,\ j=1,\ldots,n}} \|f_0\|_{L^2(\mu_0)}.$$

Taking the infimum in A, we obtain

$$E(X,\mathcal{D},\mu,\delta) \geq \sup_{\substack{f\in X\\ \|f_j\|_{L^2(\mu_j)}\leq\delta_j,\ j=1,\ldots,n}} \|f_0\|_{L^2(\mu_0)}. \tag{43}$$

Further, consider the linear space $E = L^2(\mu_1)\times\ldots\times L^2(\mu_n)$ with the semi-inner product

$$(y^1,y^2)_E = \sum_{j=1}^n \widehat{\lambda}_j (y_j^1,y_j^2)_{L^2(\mu_j)},$$

where $y^1=(y_1^1,\ldots,y_n^1)$, $y^2=(y_1^2,\ldots,y_n^2)$. Now (42) can be written in the form

$$\|\widetilde{f}-\widetilde{y}\|_E^2 \to \min,\quad f\in X,$$

where $\widetilde{f}=(f_1,\ldots,f_n)$. Since $f_{\widetilde{y}}$ is a solution of (42) it can be easily verified that for all $f\in X$

$$(\widetilde{f_{\widetilde{y}}}-\widetilde{y},\widetilde{f})_E = 0,$$

where $\widetilde{f_{\widetilde{y}}}=((f_{\widetilde{y}})_1,\ldots,(f_{\widetilde{y}})_n)$. Consequently,

$$\|\widetilde{f}-\widetilde{y}\|_E^2 = \|\widetilde{f}-\widetilde{f_{\widetilde{y}}}\|_E^2 + \|\widetilde{f_{\widetilde{y}}}-\widetilde{y}\|_E^2.$$

Thus, for all $f\in X$

$$\|\widetilde{f}-\widetilde{f_{\widetilde{y}}}\|_E^2 \leq \|\widetilde{f}-\widetilde{y}\|_E^2 = \sum_{j=1}^n \widehat{\lambda}_j \|f_j-\widetilde{y}_j\|_{L^2(\mu_j)}^2. \tag{44}$$

Let $f\in X$ and $y\in L^2(\mu_1)\times\ldots\times L^2(\mu_n)$ be such that $\|f_j-y_j\|_{L^2(\mu_j)}\leq\delta_j$, $j=1,\ldots,n$. Then for any $\varepsilon>0$ there exists $\widetilde{y}=(\widetilde{y}_1,\ldots,\widetilde{y}_n)\in Y_1\times\ldots\times Y_n$ such that $\|y_j-\widetilde{y}_j\|_{L^2(\mu_j)}<\varepsilon$, $j=1,\ldots,n$. Thus,

$$\|f_j-\widetilde{y}_j\|_{L^2(\mu_j)} \leq \|f_j-y_j\|_{L^2(\mu_j)}+\|y_j-\widetilde{y}_j\|_{L^2(\mu_j)} \leq \delta_j+\varepsilon,\quad j=1,\ldots,n.$$

Set $z=f-f_{\widetilde{y}}$. Then (44) implies that

$$\sum_{j=1}^n \widehat{\lambda}_j\|z_j\|_{L^2(\mu_j)}^2 = \|\widetilde{z}\|_E^2 \leq \sum_{j=1}^n \widehat{\lambda}_j(\delta_j+\varepsilon)^2.$$

We have the following estimate for the method $\widehat{A}$

$$\begin{aligned}\|f_0-\widehat{A}(y)\|_{L^2(\mu_0)} &\leq \|f_0-\widehat{A}(\widetilde{y})\|_{L^2(\mu_0)}+\|\widehat{A}(y-\widetilde{y})\|_{L^2(\mu_0)}\\ &\leq \|z_0\|_{L^2(\mu_0)}+\|\widehat{A}\|n\varepsilon.\end{aligned}$$

Taking into account that for all $C_1, C_2 > 0$

$$\sup_{\substack{z\in X\\ \sum_{j=1}^n \widehat{\lambda}_j \|z_j\|^2_{L^2(\mu_j)} \le C_1}} \|z_0\|^2_{L^2(\mu_0)} = \frac{C_1}{C_2} \sup_{\substack{z\in X\\ \sum_{j=1}^n \widehat{\lambda}_j \|z_j\|^2_{L^2(\mu_j)} \le C_2}} \|z_0\|^2_{L^2(\mu_0)},$$

we obtain

$$\begin{aligned}
\|f_0 - \widehat{A}(\widetilde{y})\|^2_{L^2(\mu_0)} &= \|z_0\|^2_{L^2(\mu_0)} \\
&\le \sup_{\substack{z\in X\\ \sum_{j=1}^n \widehat{\lambda}_j \|z_j\|^2_{L^2(\mu_j)} \le \sum_{j=1}^n \widehat{\lambda}_j (\delta_j+\varepsilon)^2}} \|z_0\|^2_{L^2(\mu_0)} \\
&= \frac{\sum_{j=1}^n \widehat{\lambda}_j (\delta_j + \varepsilon)^2}{\sum_{j=1}^n \widehat{\lambda}_j \delta_j^2} \sup_{\substack{z\in X\\ \sum_{j=1}^n \widehat{\lambda}_j \|z_j\|^2_{L^2(\mu_j)} \le \sum_{j=1}^n \widehat{\lambda}_j \delta_j^2}} \|z_0\|^2_{L^2(\mu_0)} \\
&= \frac{\sum_{j=1}^n \widehat{\lambda}_j (\delta_j + \varepsilon)^2}{\sum_{j=1}^n \widehat{\lambda}_j \delta_j^2} \sup_{\substack{f\in X\\ \|f_j\|_{L^2(\mu_j)} \le \delta_j,\ j=1,\dots,n}} \|f_0\|^2_{L^2(\mu_0)}.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary we obtain

$$E(X, \mathcal{D}, \mu, \delta) \le e(X, \mathcal{D}, \mu, \delta, \widehat{A}) \le \sup_{\substack{f\in X\\ \|f_j\|_{L^2(\mu_j)} \le \delta_j,\ j=1,\dots,n}} \|f_0\|_{L^2(\mu_0)}.$$

This and (43) imply

$$E(X, \mathcal{D}, \mu, \delta) = \sup_{\substack{f\in X\\ \|f_j\|_{L^2(\mu_j)} \le \delta_j,\ j=1,\dots,n}} \|f_0\|_{L^2(\mu_0)}$$

and $\widehat{A}$ is an optimal method. $\square$

We will apply Theorem 6 to the construction of optimal recovery method for the Hadamard type problem considered in section 3. Let

$$(\widehat{\lambda}_1, \widehat{\lambda}_2) = \left(\frac{r_2^2 - \rho^2}{r_2^2 - r_1^2} \left(\frac{\rho}{r_1}\right)^{2s}, \frac{\rho^2 - r_1^2}{r_2^2 - r_1^2} \left(\frac{\rho}{r_2}\right)^{2s} \right),$$

if

$$\left(\frac{r_1}{r_2}\right)^{s+1} \le \frac{\delta_1}{\delta_2} < \left(\frac{r_1}{r_2}\right)^{s}, \quad s \in \mathbb{Z}_+, \tag{45}$$

and $(\widehat{\lambda}_1, \widehat{\lambda}_2) = (0, 1)$, if $\delta_1 \geq \delta_2$. Consider the following extremal problem

$$\sup \Big\{ \|f_\rho\|_{H^2(\mathbf{B}_n)} : f \in H^2(\mathbf{B}_n),\ \widehat{\lambda}_1 \|f_{r_1}\|^2_{H^2(\mathbf{B}_n)} + \widehat{\lambda}_2 \|f_{r_2}\|^2_{H^2(\mathbf{B}_n)} \leq \widehat{\lambda}_1 \delta_1^2 + \widehat{\lambda}_2 \delta_2^2 \Big\}. \tag{46}$$

Proposition 8. *The values of extremal problems* (9) *and* (46) *are the same.*

Proof. Apply Theorem 1 to the measures $\mu_0 = ds_\rho$ and $\mu_1 = \widehat{\lambda}_1 \sigma_{r_1} + \widehat{\lambda}_2 \sigma_{r_2}$. It is possible to show the same way (11) was derived, that there exists $\nu > 0$ such that Euler's equation for problem (46) has the following form

$$-f(\rho^2 w) + \nu(\widehat{\lambda}_1 f(r_1^2 w) + \widehat{\lambda}_2 f(r_2^2 w)) = 0. \tag{47}$$

Let $\widehat{f}$ be the spectral function of problem (9) corresponding to the spectral point $(\widehat{\lambda}_1, \widehat{\lambda}_2)$. Obviously, $\widehat{f}$ is admissible for problem (46). Therefore, $\nu = 1$ belongs to the spectrum of (46). It suffices to show that there are no spectral points of (46) that are greater than one. Indeed, by Theorem 2 the value of problem (46)

$$\max\{\nu(\widehat{\lambda}_1 \delta^2 + \widehat{\lambda}_2)\},$$

where ν runs over the spectrum of (46).

Similar to the proof of Theorem 3 it may be shown that there is at least one j such that

$$-\rho^{2j} + \nu\widehat{\lambda}_1 r_1^{2j} + \nu\widehat{\lambda}_2 r_2^{2j} = 0.$$

Thus,

$$\nu = \left(\widehat{\lambda}_1 \left(\frac{r_1}{\rho} \right)^{2j} + \widehat{\lambda}_2 \left(\frac{r_2}{\rho} \right)^{2j} \right)^{-1}.$$

If $\delta_1 \geq \delta_2$, then

$$\nu = \left(\frac{\rho}{r_2} \right)^{2j} \leq 1.$$

Suppose that condition (45) is fulfilled. Then

$$\widehat{\lambda}_1 r_1^{2s} + \widehat{\lambda}_2 r_2^{2s} = \rho^{2s},$$
$$\widehat{\lambda}_1 r_1^{2(s+1)} + \widehat{\lambda}_2 r_2^{2(s+1)} = \rho^{2(s+1)}.$$

Consider the function

$$\chi(t) = \widehat{\lambda}_1 e^{at} + \widehat{\lambda}_2 e^{bt},$$

where

$$a = \log \frac{r_1}{\rho}, \quad b = \log \frac{r_2}{\rho}.$$

We have

$$\chi(s) = \chi(s+1) = 1.$$

Since χ is a convex function for all $j \in \mathbb{Z}_+$, $\chi(j) \geq 1$. It means that $\nu \leq 1$. $\square$

Theorem 7. *Let* $(\widehat{\lambda}_1, \widehat{\lambda}_2)$ *be the same as in problem* (46). *Then the error of optimal recovery is given by*

$$\sqrt{\widehat{\lambda}_1 \delta_1^2 + \widehat{\lambda}_2 \delta_2^2}$$

and the method

$$\widehat{A}(y_1, y_2)(z) = \sum_{j=0}^{\infty} \frac{1}{\widehat{\lambda}_1 r_1^{2j} + \widehat{\lambda}_2 r_2^{2j}} \sum_{|\alpha|=j} (\widehat{\lambda}_1 r_1^{2j} c_\alpha^{(1)} + \widehat{\lambda}_2 r_2^{2j} c_\alpha^{(2)}) z^\alpha,$$

where

$$c_\alpha^{(k)} = \frac{(n+|\alpha|-1)!}{n!\alpha! r_k^{|\alpha|}} \langle y_k(r_k z), z^\alpha \rangle_{L_2(\sigma)}, \quad k = 1, 2,$$

is optimal.

Proof. Let $y_1(r_1 z)$ and $y(r_2 z)$ be arbitrary functions from $L_2(\sigma)$. Consider the extremal problem

$$\widehat{\lambda}_1 \|f(r_1 z) - y_1(r_1 z)\|_{L_2(\sigma)}^2 + \widehat{\lambda}_2 \|f(r_2 z) - y_2(r_2 z)\|_{L_2(\sigma)}^2 \to \min, \quad f \in H^2(\mathbf{B}_n). \tag{48}$$

Write $y_k(r_k z)$, $k = 1, 2$,

$$y_k(r_k z) = \sum_{j=0}^{\infty} r_k^j \sum_{\substack{|\alpha|=j \\ \alpha \geq 0}} c_\alpha^{(k)} z^\alpha + \widetilde{y}_k(z),$$

where $\widetilde{y}_k$ are orthogonal to all holomorphic polynomials in $L_2(\sigma)$. Then problem (48) may be written in the form

$$\widehat{\lambda}_1 \sum_{j=0}^{\infty} \frac{r_1^{2j} n!}{(n+j-1)!} \sum_{|\alpha|=j} \alpha! |f_\alpha - c_\alpha^{(1)}|^2 + \|\widetilde{y}_1\|_{L_2(\sigma)}^2$$
$$+ \widehat{\lambda}_2 \sum_{j=0}^{\infty} \frac{r_2^{2j} n!}{(n+j-1)!} \sum_{|\alpha|=j} \alpha! |f_\alpha - c_\alpha^{(2)}|^2 + \|\widetilde{y}_2\|_{L_2(\sigma)}^2 \to \min,$$
$$f \in H^2(\mathbf{B}_n).$$

It is easy to show that for all functions $y_1(r_1z), y_2(r_2z) \in L_2(\sigma)$ with finite number of coefficients $c_\alpha^{(k)} \neq 0$, $k = 1, 2$, the solution of this problem is

$$f(z) = \sum_{j=0}^{\infty} \frac{1}{\widehat{\lambda}_1 r_1^{2j} + \widehat{\lambda}_2 r_2^{2j}} \sum_{|\alpha|=j} (\widehat{\lambda}_1 r_1^{2j} c_\alpha^{(1)} + \widehat{\lambda}_2 r_2^{2j} c_\alpha^{(2)}) z^\alpha.$$

Since such functions are dense in $L_2(\sigma)$ the required statement follows from Theorem 6. □

Let us turn to the optimal recovery method corresponding to the generalized Schwarz lemma. Consider the extremal problem

$$\sup \left\{ \int_\Gamma |f|^2 \, d\mu : f \in H^2, \ \widehat{\lambda}_1 |f(a)|^2 + \widehat{\lambda}_2 \|f\|_{H^2}^2 \leq \widehat{\lambda}_1 \delta^2 + \widehat{\lambda}_2 \right\}, \tag{49}$$

where as before μ is the normalized Lebesgue measure on Γ and $(\widehat{\lambda}_1, \widehat{\lambda}_2)$ is an extremal spectral point for problem (17).

Proposition 9. *Suppose that either* $a \neq b$ *and* $\delta \leq |\varphi_1(a)|$, *or* $|\varphi_1(a)| < \delta < |\varphi_0(a)|$ *and* $\gamma = \left| \dfrac{a-b}{1-ab} \right| \geq b^{2/3}$. *Then the values of extremal problems* (17) *and* (49) *are the same.*

Proof. The same argument as in the proof of Proposition 8 shows that there is a positive ν such that the Euler equation for problem (17) is

$$\frac{1}{1-\rho w} f\left(\frac{\rho}{1-\rho w}\right) = \nu \left(\widehat{\lambda}_1 \frac{f(a)}{1-\bar{a}w} + \widehat{\lambda}_2 f(w) \right). \tag{50}$$

Also similarly to the proof of Proposition 8 it suffices to prove that there are no spectral points of (49) that are greater than one.

Now, let ν belong to the spectrum of (49). First, let us show that a function which satisfies (50) does not vanish at a. Indeed, let $f(a) = 0$. Then f is an eigenfunction of operator T. As it was shown above, functions (19) are the only eigenfunctions of T. Since these functions may vanish only at b and $b \neq a$, we come to a contradiction.

Since $f(a) \neq 0$, the argument in the proof of Theorem 4 (equation (27)) shows that

$$\nu \widehat{\lambda}_1 = h(\nu \widehat{\lambda}_2). \tag{51}$$

Propositions 6 and 7 imply that $\zeta_0 < \widehat{\lambda}_2 < \alpha_0$. Note that the function h decreases on the interval (ζ_0, α_0) and is negative for $\lambda > \alpha_0$. If we suppose that $\nu > 1$, then $h(\nu\widehat{\lambda}_2)$ either negative or strictly less than $\widehat{\lambda}_1$. The contradiction shows that $\nu \leq 1$. □

Proposition 10. *If $a = b$ and $\delta < \varphi(b) = 1/\sqrt{1-b^2}$, then the values of extremal problems* (17) *and* (49) *are the same.*

Proof. The equation (50) is still satisfied with $a = b$. The same argument as the one in the proof of Proposition 9 shows that here it also suffices to prove that $\nu \le 1$. By Proposition 3

$$(\widehat{\lambda}_1, \widehat{\lambda}_2) = ((1-b^2)(\alpha_0 - \alpha_1), \alpha_1). \tag{52}$$

If $f(b) = 0$, then (50) implies that f is an eigenfunction of T with eigenvalue $\nu\alpha_1$. Since $\varphi_0(b) \neq 0$, we have $\nu\alpha_1 = \alpha_j$, $j \ge 1$. Consequently, $\nu \le 1$.

If $f(b) \neq 0$, then (51) is satisfied. In our case it turns into

$$\nu\widehat{\lambda}_1 = (\alpha_0 - \nu\widehat{\lambda}_2)(1-b^2).$$

Substituting $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$ from (52), we obtain that $\nu = 1$. □

Theorem 8. *Suppose that one of the following conditions is satisfied*

1. $\delta \ge |\varphi_0(a)|$,
2. $\delta \le |\varphi_1(a)|$,
3. $|\varphi_1(a)| < \delta < |\varphi_0(a)|$, $\gamma \ge b^{2/3}$,
4. $a = b$,

and $(\widehat{\lambda}_1, \widehat{\lambda}_2)$ the corresponding extremal spectral point. Then the error of optimal recovery is given by

$$\sqrt{\widehat{\lambda}_1\delta^2 + \widehat{\lambda}_2}$$

and the method

$$\widehat{A}(y)(z) = \frac{\widehat{\lambda}_1 y}{\widehat{\lambda}_1 + \widehat{\lambda}_2(1-|a|^2)} \frac{1-|a|^2}{1-\overline{a}z} \tag{53}$$

is optimal.

Proof. For an arbitrary $y \in \mathbb{C}$ consider the extremal problem

$$\widehat{\lambda}_1|f(a) - y|^2 + \widehat{\lambda}_2\|f\|^2_{H^2} \to \min, \quad f \in H^2. \tag{54}$$

Using the representation

$$f(z) = \sum_{j=0}^{\infty} c_j\psi_j(z),$$

where

$$\psi_j(z) = \frac{\sqrt{1-|a|^2}}{1-\overline{a}z}\left(\frac{a-z}{1-\overline{a}z}\right)^j,$$

problem (54) may be written in the form

$$\widehat{\lambda}_1 \left| \frac{c_0}{\sqrt{1-|a|^2}} - y \right|^2 + \widehat{\lambda}_2 \sum_{j=0}^{\infty} |c_j|^2 \to \min, \quad f \in H^2.$$

It is easy to see that the solution of this problem is

$$f(z) = \frac{\widehat{\lambda}_1 y}{\widehat{\lambda}_1 + \widehat{\lambda}_2(1-|a|^2)} \frac{1-|a|^2}{1-\overline{a}z}.$$

By Propositions 2, 9, and 10, Theorem 6 is applicable and the required statement immediately follows from it. □

Note that for $a = b$ the optimal method of recovery (53) does not depend on δ and has the form

$$\widehat{A}(y)(z) = \frac{(1-b^2)^2}{1-bz}.$$

5. Concluding Remarks

In this section we would like to discuss several open problems related to the results obtained above.

1. Our first problem is related to Theorem 6. This theorem gives an effective way of constructing optimal recovery algorithms. Unfortunately, every time it is necessary to verify whether the values of the dual extremal problem and the problem with a single "mixed" constraint are the same. It would be convenient to have some general condition under which this coincidence takes place for $n > 2$ (for $n = 2$ see [7]).

2. Returning to the Generalized Schwarz Lemma, problem (17), it would be desirable to identify the extremal spectral point in all possible cases. We have shown that in a number of cases the extremal spectral point is the only point in Λ_2 such that $\zeta_0 < \widehat{\lambda}_2 < \alpha_0$. Our attempts to find a nontrivial-case when this point is not extremal failed. Thus, we are tempted to conjecture that the point of Λ_2 with the biggest λ_2 is always extremal.

Conjecture. *If $a \neq b$ and $\delta < |\varphi_0(a)|$, the point in Λ_2 such that $\zeta_0 < \widehat{\lambda}_2 < \alpha_0$ is always the spectral extremal point for problem* (17).

3. It is natural to ask which choice of a minimizes the value of problem (17) (of course, this choice of a leads to the least optimal recovery error). It follows from above discussion that the point b plays a special role.

Problem. *Does the choice $a = b$ always lead to the least mean square optimal recovery error?*

4. Finally, if in problem (17) we replace the constraint $|f(a)| \leq \delta$ with

$$\frac{1}{2\pi r} \int_{|z-a|=r} |f(z)|^2 |d(z-a)| \leq \delta, \ (0 < r < 1 - |a|),$$

then the problem becomes even more difficult. The reason is that in the right hand side of Euler's equation the term $\lambda_1 \dfrac{f(a)}{1 - \bar{a}z}$ is replaced with

$$\frac{\lambda_1}{1 - \bar{a}z} f\left(a - \frac{r^2 z}{1 - \bar{a}z}\right).$$

Thus, finding the spectrum in this case is reduced to finding eigenvalues of an operator which is a linear combination of two compact non-commuting operatoprs.

It would be very interesting to find the eigenbasis which corresponds to this problem and to find the solution.

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