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Optimal recovery of operators and multidimensional Carlson type inequalities

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ABSTRACT

The paper is concerned with recovery problems of linear multiplier-type operators from noisy information on weighted classes of functions. Optimal methods of recovery are constructed. The dual extremal problem is closely connected with Carlson type inequalities.

1. General setting

Let T be a nonempty set, Σ be the σ-algebra of subsets of T, and μ be a nonnegative σ-additive measure on Σ. We denote by $L_p(T, \Sigma, \mu)$ (or simply $L_p(T, \mu)$) the set of all Σ-measurable functions with values in $\mathbb{R}$ or in $\mathbb{C}$ for which

$$\|x(\cdot)\|_{L_p(T,\mu)} = \left(\int_T |x(t)|^p \, d\mu(t)\right)^{1/p} < \infty, \quad 1 \le p < \infty,$$
$$\|x(\cdot)\|_{L_\infty(T,\mu)} = \operatorname*{ess\,sup}_{t\in T} |x(t)| < \infty, \quad p = \infty.$$

Put

$$\mathcal{W} = \{x(\cdot) \in L_p(T, \mu) : \|\varphi(\cdot)x(\cdot)\|_{L_r(T,\mu)} < \infty\},$$
$$W = \{x(\cdot) \in \mathcal{W} : \|\varphi(\cdot)x(\cdot)\|_{L_r(T,\mu)} \le 1\},$$

where $1 \le p, r \le \infty$, and $\varphi(\cdot)$ is a measurable function on T. Consider the problem of recovery of operator $\Lambda\colon \mathcal{W} \to L_q(T, \mu)$, $1 \le q \le \infty$, defined by equality $\Lambda x(\cdot) = \psi(\cdot)x(\cdot)$, where $\psi(\cdot)$

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is a measurable function on T, on the class W by the information about functions $x(\cdot) \in W$ given inaccurately. More precisely, we assume that for any function $x(\cdot) \in W$ we know $y(\cdot) \in L_p(T_0, \mu)$, where T_0 is not empty μ-measurable subset of T, such that $\|x(\cdot) - y(\cdot)\|_{L_p(T_0,\mu)} \le \delta$, $\delta \ge 0$. We want to approximate the value $\Lambda x(\cdot)$ knowing $y(\cdot)$.

As recovery methods we consider all possible mappings

$$m\colon L_p(T_0, \mu) \to L_q(T, \mu).$$

The error of a method m is defined as

$$e(p, q, r, m) = \sup_{\substack{x(\cdot)\in W,\ y(\cdot)\in L_p(T_0,\mu)\\ \|x(\cdot)-y(\cdot)\|_{L_p(T_0,\mu)}\le\delta}} \|\Lambda x(\cdot) - m(y)(\cdot)\|_{L_q(T,\mu)}.$$

The quantity

$$E(p, q, r) = \inf_{m\colon L_p(T_0,\mu)\to L_q(T,\mu)} e(p, q, r, m) \tag{1}$$

is known as the optimal recovery error, and a method on which this infimum is attained is called optimal.

Various settings of optimal recovery theory and examples of such problems may be found in [11,12,17,18,15,13]. Much of them are devoted to optimal recovery of linear functionals. There are not so many results about optimal recovery of linear operators when non-Euclidean metrics is used [12, Theorem 12 on p. 45], [6,14]. In [14] we considered problem (1) when any two of p, q, and r coincide. Here we analyze the case when all metrics can be different and $1 \le q < p, r < \infty$. We construct optimal method of recovery, find its error, and apply this result to obtain exact constants in Carlson type inequalities. The case $p = \infty$ and/or $r = \infty$ requires a slightly different approach. Some particular results of such kind may be found in [8] ($T = \mathbb{Z}$) and [9] ($T = \mathbb{R}$).

2. Main results

Let $\chi_0(\cdot)$ be the characteristic function of the set T_0:

$$\chi_0(t) = \begin{cases} 1, & t \in T_0, \\ 0, & t \notin T_0. \end{cases}$$

Theorem 1. *Let $1 \le q < p, r < \infty$, $\lambda_1, \lambda_2 \ge 0$, $\lambda_1 + \lambda_2 > 0$, $\varphi(t) \ne 0$ for almost all $t \in T \setminus T_0$, $\widehat{x}(t) = \widehat{x}(t, \lambda_1, \lambda_2) \ge 0$ be a solution of equation*

$$-q|\psi(t)|^q + p\lambda_1 x^{p-q}(t)\chi_0(t) + r\lambda_2|\varphi(t)|^r x^{r-q}(t) = 0, \tag{2}$$

and λ_1, λ_2 such that

$$\begin{aligned} &\int_{T_0} \widehat{x}^p(t)\,d\mu(t) \le \delta^p, && \int_T |\varphi(t)|^r \widehat{x}^r(t)\,d\mu(t) \le 1, \\ &\lambda_1\left(\int_{T_0} \widehat{x}^p(t)\,d\mu(t) - \delta^p\right) = 0, && \lambda_2\left(\int_T |\varphi(t)|^r \widehat{x}^r(t)\,d\mu(t) - 1\right) = 0, \end{aligned} \tag{3}$$

and $\lambda_2 > 0$, if $T \setminus T_0 \ne \emptyset$. Then

$$E(p, q, r) = \left(\frac{p\lambda_1\delta^p + r\lambda_2}{q}\right)^{1/q},$$

and the method

$$\widehat{m}(y)(t) = \begin{cases} q^{-1}p\lambda_1\widehat{x}^{p-q}(t)|\psi(t)|^{-q}\psi(t)y(t), & t \in T_0,\ \psi(t) \ne 0, \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

is optimal recovery method.

To prove this theorem we need some preliminary results.

Lemma 1.

$$E(p,q,r) \geq \sup_{\substack{x(\cdot)\in W \\ \|x(\cdot)\|_{L_p(T_0,\mu)}\leq\delta}} \|\Lambda x(\cdot)\|_{L_q(T,\mu)}. \tag{5}$$

The lower bound of type (5) is the well-known result which is usually applied to obtain the error of optimal recovery. In more or less general forms it was proved in many papers (see, for example, [14]).

The extremal problem which arises on the right-hand side of (5), known as the dual problem, may be written as

$$\begin{aligned} &\|\psi(\cdot)x(\cdot)\|_{L_q(T,\mu)} \to \max, \quad \|x(\cdot)\|_{L_p(T_0,\mu)} \leq \delta, \\ &\|\varphi(\cdot)x(\cdot)\|_{L_r(T,\mu)} \leq 1. \end{aligned} \tag{6}$$

For $T_0 = T \subset \mathbb{R}^n$ and $q = 1$ problem (6) was examined in [2] in connection with Stechkin's problem.

We give a straightforward result (resembling the sufficient conditions in the Kuhn–Tucker theorem), which we will require in solving dual problems similar to (6).

Let $f_j : A \to \mathbb{R}$, $j = 0, 1, \ldots, n$, be functions defined on some set A. Consider the extremal problem

$$f_0(x) \to \max, \qquad f_j(x) \leq 0, \quad j = 1, \ldots, n, \; x \in A, \tag{7}$$

and write down its Lagrange function

$$\mathcal{L}(x,\lambda) = -f_0(x) + \sum_{j=1}^{n} \lambda_j f_j(x), \quad \lambda = (\lambda_1, \ldots, \lambda_n).$$

Lemma 2 ([14]). *Assume that there exist $\widehat{\lambda}_j \geq 0$, $j = 1, \ldots, n$, and an element $\widehat{x} \in A$, admissible for problem (7), such that*

(a) $\min\limits_{x\in A} \mathcal{L}(x,\widehat{\lambda}) = \mathcal{L}(\widehat{x},\widehat{\lambda})$, $\quad \widehat{\lambda} = (\widehat{\lambda}_1, \ldots, \widehat{\lambda}_n)$,

(b) $\sum\limits_{j=1}^{n} \widehat{\lambda}_j f_j(\widehat{x}) = 0.$

Then $\widehat{x}$ is an extremal element for problem (7).

Put

$$F(u,v,\alpha) = -((1-\alpha)u + \alpha v)^q + av^p + bu^r, \quad u, v \geq 0, \; \alpha \in [0,1],$$

where $a, b \geq 0$, and $1 \leq p, q, r < \infty$.

Lemma 3. *For all $a, b \geq 0$, $a + b > 0$, and all $1 \leq q < p, r < \infty$, there exists the unique solution $\widehat{u} > 0$ of the equation*

$$-q + pau^{p-q} + rbu^{r-q} = 0. \tag{8}$$

Moreover, for all $u, v \geq 0$ and $\alpha = q^{-1}pa\widehat{u}^{p-q} = 1 - q^{-1}rb\widehat{u}^{r-q}$

$$F(\widehat{u},\widehat{u},\alpha) \leq F(u,v,\alpha). \tag{9}$$

In particular, for all $u \geq 0$

$$-\widehat{u}^q + a\widehat{u}^p + b\widehat{u}^r \leq -u^q + au^p + bu^r.$$

Proof. The existence of the unique solution of (8) follows from the fact that the continuous function $f(u) = pau^{p-q} + rbu^{r-q}$ increases monotonically from 0 to $+\infty$.

Let us prove (9). The cases $a = 0$ or $b = 0$ are easily obtained by finding the minimum of $F(u, v, 0) = -u^q + bu^r$ if $a = 0$ or $F(u, v, 1) = -v^q + av^p$ if $b = 0$. Assume that $a, b > 0$. Then $\alpha \in (0, 1)$. Let

$$C > \max\{a^{-\frac{1}{p-q}}, b^{-\frac{1}{r-q}}\}.$$

Then for $u \geq C$ and $v \leq u$ we have

$$F(u, v, \alpha) \geq -u^q + bu^r = u^q(-1 + bu^{r-q}) > 0. \tag{10}$$

If $v \geq C$ and $v \geq u$, then

$$F(u, v, \alpha) \geq -v^q + av^p = v^q(-1 + av^{p-q}) > 0. \tag{11}$$

Since $F(0, 0, \alpha) = 0$ we obtain that

$$\inf_{(u,v)\in\mathbb{R}^2_+} F(u, v, \alpha) = \inf_{\substack{0\leq u\leq C\\ 0\leq v\leq C}} F(u, v, \alpha).$$

It follows from the Weierstrass extreme value theorem that there exist $0 \leq u_0 \leq C$ and $0 \leq v_0 \leq C$ such that

$$\inf_{(u,v)\in\mathbb{R}^2_+} F(u, v, \alpha) = F(u_0, v_0, \alpha).$$

In view of (10) and (11) $u_0 < C$ and $v_0 < C$. We have

$$\begin{aligned} F_u(u, v, \alpha) &= -q((1-\alpha)u + \alpha v)^{q-1}(1-\alpha) + rbu^{r-1} \\ &= rb(-((1-\alpha)u + \alpha v)^{q-1}\widehat{u}^{r-q} + u^{r-1}). \end{aligned}$$

Thus, for any $v_0 \geq 0$ and sufficiently small $u > 0$ $F_u(u, v_0, \alpha) < 0$. Consequently,

$$F(u, v_0, \alpha) < F(0, v_0, \alpha)$$

for sufficiently small u. It means that $0 < u_0 < C$. The similar arguments show that $0 < v_0 < C$. Hence

$$F_u(u_0, v_0, \alpha) = F_v(u_0, v_0, \alpha) = 0.$$

Since

$$\begin{aligned} F_v(u, v, \alpha) &= -q((1-\alpha)u + \alpha v)^{q-1}\alpha + pav^{p-1} \\ &= pa(-((1-\alpha)u + \alpha v)^{q-1}\widehat{u}^{p-q} + v^{p-1}) \end{aligned}$$

we have

$$-((1-\alpha)u_0 + \alpha v_0)^{q-1}\widehat{u}^{r-q} + u_0^{r-1} = 0, \tag{12}$$

$$-((1-\alpha)u_0 + \alpha v_0)^{q-1}\widehat{u}^{p-q} + v_0^{p-1} = 0. \tag{13}$$

Consequently,

$$\frac{u_0^{r-1}}{v_0^{p-1}} = \widehat{u}^{r-p}.$$

Suppose that $p \leq r$. Substituting

$$u_0 = \widehat{u}^{\frac{r-p}{r-1}} v_0^{\frac{p-1}{r-1}} \tag{14}$$

into (13), we obtain the equality

$$(\alpha v_0 + (1-\alpha)\widehat{u}^{\frac{r-p}{r-1}} v_0^{\frac{p-1}{r-1}})^{q-1}\widehat{u}^{p-q} = v_0^{p-1}.$$

This equality may be rewritten in the form

$$(\alpha + (1-\alpha)t^{\frac{p-r}{r-1}})^{q-1} = t^{p-q}, \tag{15}$$

where $t = v_0\widehat{u}^{-1}$. It is easily seen that (15) has the unique solution $t = 1$. Consequently, $v_0 = \widehat{u}$ and it follows by (14) that $u_0 = \widehat{u}$.

If $p > r$, then we substitute

$$v_0 = \widehat{u}^{\frac{p-r}{p-1}} u_0^{\frac{r-1}{p-1}}$$

into (12). Similar to the previous case we obtain the equality which may be written in the form

$$(\alpha s^{\frac{r-p}{p-1}} + 1 - \alpha)^{q-1} = s^{r-q}, \tag{16}$$

where $s = u_0\widehat{u}^{-1}$. The unique solution of (16) is $s = 1$. Thus, for the case when $p > r$ we have the same solution of (12), (13) $u_0 = v_0 = \widehat{u}$. Hence, for all $u, v \geq 0$

$$F(u, v, \alpha) \geq \inf_{(u,v)\in\mathbb{R}^2_+} F(u, v, \alpha) = F(\widehat{u}, \widehat{u}, \alpha). \quad \square$$

Proof of Theorem 1. 1. Lower estimate. The extremal problem (6) (for convenience, we raise the quantity to be maximized to the qth power) is as follows:

$$\int_T |\psi(t)x(t)|^q \, d\mu(t) \to \max, \qquad \int_{T_0} |x(t)|^p \, d\mu(t) \leq \delta^p,$$
$$\int_T |\varphi(t)x(t)|^r \, d\mu(t) \leq 1. \tag{17}$$

The Lagrange function for this problem reads as

$$\mathcal{L}(x(\cdot), \lambda_1, \lambda_2) = \int_T L\big(t, x(t), \lambda_1, \lambda_2\big) \, d\mu(t),$$

where

$$L(t, x, \lambda_1, \lambda_2) = -|\psi(t)x|^q + \lambda_1|x|^p\chi_0(t) + \lambda_2|\varphi(t)x|^r.$$

If $t \in T$ such that $\psi(t) = 0$, then evidently $\widehat{x}(t) = 0$ and for those t for all $x(\cdot) \in \mathcal{W}$

$$L(t, 0, \lambda_1, \lambda_2) \leq L(t, x(t), \lambda_1, \lambda_2).$$

Using this fact and Lemma 3, we obtain that there is the unique solution $\widehat{x}(\cdot)$ of (2) and, moreover, for almost all $t \in T$ and all $x(\cdot) \in \mathcal{W}$

$$L(t, \widehat{x}(t), \lambda_1, \lambda_2) \leq L(t, x(t), \lambda_1, \lambda_2).$$

Consequently,

$$\mathcal{L}(\widehat{x}(\cdot), \lambda_1, \lambda_2) \leq \mathcal{L}(x(\cdot), \lambda_1, \lambda_2).$$

Taking into account (3) we obtain by Lemma 2 that $\widehat{x}(\cdot)$ is the extremal function in (17). It follows by (5) that

$$E(p, q, r) \geq \left(\int_T |\psi(t)|^q \widehat{x}^q(t) \, d\mu(t)\right)^{1/q}.$$

From (2) we have

$$|\psi(t)|^q \widehat{x}^q(t) = q^{-1}p\lambda_1\widehat{x}^p(t)\chi_{T_0}(t) + q^{-1}r\lambda_2|\varphi(t)|^r\widehat{x}^r(t).$$

Integrating this equality over the set T, we obtain

$$\int_T |\psi(t)|^q \widehat{x}^q(t)\, d\mu(t) = \frac{p\lambda_1\delta^p + r\lambda_2}{q}. \tag{18}$$

Thus,

$$E(p,q,r) \ge \left(\frac{p\lambda_1\delta^p + r\lambda_2}{q}\right)^{1/q}.$$

2. Upper estimate. To estimate the error of method (4) we need to find the value of the extremal problem:

$$\begin{gathered}\int_{T_0} |\psi(t)x(t) - \psi(t)\alpha(t)y(t)|^q\, d\mu(t) + \int_{T\setminus T_0} |\psi(t)x(t)|^q\, d\mu(t) \to \max,\\ \int_{T_0} |x(t)-y(t)|^p\, d\mu(t) \le \delta^p, \quad \int_T |\varphi(t)x(t)|^r\, d\mu(t) \le 1,\end{gathered} \tag{19}$$

where

$$\alpha(t) = \begin{cases} q^{-1}p\lambda_1\widehat{x}^{p-q}(t)|\psi(t)|^{-q}, & t\in T_0,\ \psi(t)\ne 0,\\ 0, & \text{otherwise.}\end{cases} \tag{20}$$

Taking

$$z(t) = \begin{cases} x(t)-y(t), & t\in T_0,\\ 0, & t\in T\setminus T_0,\end{cases}$$

we rewrite (19) as follows:

$$\begin{gathered}\int_T |\psi(t)|^q |(1-\alpha(t))x(t) + \alpha(t)z(t)|^q\, d\mu(t) \to \max,\\ \int_{T_0} |z(t)|^p\, d\mu(t) \le \delta^p, \quad \int_T |\varphi(t)x(t)|^r\, d\mu(t) \le 1.\end{gathered}$$

The value of this problem does not exceed the value of the problem

$$\begin{gathered}\int_T |\psi(t)|^q ((1-\alpha(t))u(t) + \alpha(t)v(t))^q\, d\mu(t) \to \max,\\ \int_{T_0} v^p(t)\, d\mu(t) \le \delta^p, \qquad \int_T |\varphi(t)|^r u^r(t)\, d\mu(t) \le 1,\\ u(t)\ge 0,\ v(t)\ge 0 \quad \text{for almost all } t\in T.\end{gathered} \tag{21}$$

The Lagrange function for this problem is

$$\mathcal{L}_1(u(\cdot), v(\cdot), \mu_1, \mu_2) = \int_T L_1(t, u(t), v(t), \mu_1, \mu_2)\, d\mu(t),$$

where

$$\begin{aligned} L_1(t,u,v,\mu_1,\mu_2) &= -|\psi(t)|^q((1-\alpha(t))u + \alpha(t)v)^q\\ &\quad + \mu_1 v^p \chi_0(t) + \mu_2|\varphi(t)|^r u^r.\end{aligned}$$

By Lemma 3 we have

$$L_1(t, \widehat{x}(t), \widehat{x}(t), \lambda_1, \lambda_2) \le L_1(t, u(t), v(t), \lambda_1, \lambda_2).$$

Thus,

$$\mathcal{L}_1(\widehat{x}(\cdot), \widehat{x}(\cdot), \lambda_1, \lambda_2) \le \mathcal{L}_1(u(\cdot), v(\cdot), \lambda_1, \lambda_2).$$

It follows by Lemma 2 that functions $u(t) = v(t) = \widehat{x}(t)$ are extremal in (21). Consequently,

$$e(p, q, r, \widehat{m}) \le \left(\int_T |\psi(t)|^q \widehat{x}^q(t)\, d\mu(t)\right)^{1/q} = \left(\frac{p\lambda_1\delta^p + r\lambda_2}{q}\right)^{1/q} \le E(p, q, r).$$

It means that the method (4) is optimal and the optimal recovery error is as stated. □

Note that if conditions of Theorem 1 hold we proved the equality

$$E(p, q, r) = \sup_{\substack{\|x(\cdot)\|_{L_p(T_0,\mu)} \le \delta \\ \|\varphi(\cdot)x(\cdot)\|_{L_r(T,\mu)} \le 1}} \|\psi(\cdot)x(\cdot)\|_{L_q(T,\mu)}. \tag{22}$$

Corollary 1. *Let $1 \le q < p, r < \infty$, $\varphi(t) \ne 0$ for almost all $t \in T$, and*

$$0 < \int_T \left|\frac{\psi(t)}{\varphi(t)}\right|^{\frac{qr}{r-q}} d\mu(t) < \infty, \qquad \int_{T_0} \left(\frac{|\psi(t)|^q}{|\varphi(t)|^r}\right)^{\frac{p}{r-q}} d\mu(t) < \infty.$$

Then for all

$$\delta \ge \frac{\left(\int_{T_0} \left(\frac{|\psi(t)|^q}{|\varphi(t)|^r}\right)^{\frac{p}{r-q}} d\mu(t)\right)^{1/p}}{\left(\int_T \left|\frac{\psi(t)}{\varphi(t)}\right|^{\frac{qr}{r-q}} d\mu(t)\right)^{1/r}}$$

$$E(p, q, r) = \left(\int_T \left|\frac{\psi(t)}{\varphi(t)}\right|^{\frac{qr}{r-q}} d\mu(t)\right)^{\frac{r-q}{qr}},$$

and the method $\widehat{m}(y)(t) = 0$ is optimal recovery method.

Proof. It suffices to check that $\lambda_1 = 0$ and

$$\lambda_2 = \frac{q}{r}\left(\int_T \left|\frac{\psi(t)}{\varphi(t)}\right|^{\frac{qr}{r-q}} d\mu(t)\right)^{\frac{r-q}{r}}$$

satisfy the conditions of Theorem 1. □

Corollary 2. *Let $1 \le q < p, r < \infty$, $T_0 = T$, and*

$$0 < \int_T |\varphi(t)|^r |\psi(t)|^{\frac{qr}{p-q}} d\mu(t) < \infty, \qquad \int_T |\psi(t)|^{\frac{qp}{p-q}} d\mu(t) < \infty.$$

Then for all

$$\delta \le \frac{\left(\int_T |\psi(t)|^{\frac{qp}{p-q}} d\mu(t)\right)^{1/p}}{\left(\int_T |\varphi(t)|^r |\psi(t)|^{\frac{qr}{p-q}} d\mu(t)\right)^{1/r}}$$

$$E(p, q, r) = \delta\left(\int_T |\psi(t)|^{\frac{qp}{p-q}} d\mu(t)\right)^{\frac{p-q}{qp}},$$

and the method $\widehat{m}(y)(t) = \psi(t)y(t)$ is optimal recovery method.

Proof. It suffices to check that

$$\lambda_1 = \frac{q}{p\delta^{p-q}}\left(\int_T |\psi(t)|^{\frac{qp}{p-q}} d\mu(t)\right)^{\frac{p-q}{p}}$$

and $\lambda_2 = 0$ satisfy the conditions of Theorem 1. □

Note that assumption (3) need not be satisfied in all cases. For example, in the trivial case $\delta = 0$, $T_0 = T$, and $\psi(t) = 1$ there are no such λ_1 and λ_2 which satisfy (3).

Let us consider the problem of optimal recovery of the linear functional

$$Lx = \int_T \psi(t)x(t)\,d\mu(t)$$

on the class W, knowing $y(\cdot) \in L_p(T_0,\mu)$, $T_0 \subset T$, such that $\|x(\cdot) - y(\cdot)\|_{L_p(T_0,\mu)} \le \delta$, $\delta \ge 0$. In this case as recovery methods we consider all possible mappings $m\colon L_p(T_0,\mu) \to \mathbb{C}$ or $\mathbb{R}$. The error of a method m is defined as

$$e_1(p,r,m) = \sup_{\substack{x(\cdot)\in W,\ y(\cdot)\in L_p(T_0,\mu)\\ \|x(\cdot)-y(\cdot)\|_{L_p(T_0,\mu)}\le\delta}} |Lx - m(y)|.$$

The quantity

$$E_1(p,r) = \inf_{m\colon L_p(T_0,\mu)\to\mathbb{C}(\mathbb{R})} e_1(q,r,m) \tag{23}$$

is optimal recovery error, and a method on which this infimum is attained is called optimal.

Theorem 1′. *Let $1 < p, r < \infty$, $\lambda_1, \lambda_2 \ge 0$, $\lambda_1 + \lambda_2 > 0$, $\varphi(t) \ne 0$ for almost all $t \in T \setminus T_0$, $\widehat{x}(t) = \widehat{x}(t,\lambda_1,\lambda_2) \ge 0$ be a solution of equation*

$$-|\psi(t)| + p\lambda_1 x^{p-1}(t)\chi_0(t) + r\lambda_2|\varphi(t)|^r x^{r-1}(t) = 0,$$

and λ_1, λ_2 such that conditions (3) are fulfilled, and $\lambda_2 > 0$, if $T \setminus T_0 \ne \emptyset$. Then

$$E_1(p,r) = p\lambda_1\delta^p + r\lambda_2,$$

and the method

$$\widehat{m}(y) = p\lambda_1 \int_{T_0} \widehat{x}^{p-1}(t)\varepsilon(t)y(t)\,d\mu(t), \tag{24}$$

where

$$\varepsilon(t) = \begin{cases} \dfrac{\psi(t)}{|\psi(t)|}, & \psi(t) \ne 0,\\ 1, & \psi(t) = 0, \end{cases}$$

is optimal recovery method.

Proof. For the functional case it is known (see, for example, [7]) that

$$E_1(p,r) = \sup_{\substack{x(\cdot)\in W\\ \|x(\cdot)\|_{L_p(T_0,\mu)}\le\delta}} \left|\int_T \psi(t)x(t)\,d\mu(t)\right|.$$

Put $\widetilde{x}(\cdot) = \overline{\varepsilon(\cdot)}\widehat{x}(\cdot)$. It follows by (3) that $\widetilde{x}(\cdot) \in W$ and $\|\widetilde{x}(\cdot)\|_{L_p(T_0,\mu)} \le \delta$. Taking into account (18), we obtain

$$E_1(p,r) \ge \left|\int_T \psi(t)\widetilde{x}(t)\,d\mu(t)\right| = \int_T |\psi(t)|\widehat{x}(t)\,d\mu(t) = p\lambda_1\delta^p + r\lambda_2.$$

Now we estimate the error of method (24). We have

$$\begin{aligned} e_1(p,r,\widehat{m}) &= \sup_{\substack{x(\cdot)\in W,\ y(\cdot)\in L_p(T_0,\mu)\\ \|x(\cdot)-y(\cdot)\|_{L_p(T_0,\mu)}\le\delta}} \left|\int_T \psi(t)x(t)\,d\mu(t) - \widehat{m}(y)\right| \\ &\le \sup_{\substack{x(\cdot)\in W,\ z(\cdot)\in L_p(T_0,\mu)\\ \|z(\cdot)\|_{L_p(T_0,\mu)}\le\delta}} \int_T |\psi(t)||(1-\alpha(t))x(t) + \alpha(t)z(t)|\,d\mu(t), \end{aligned}$$

where $\alpha(\cdot)$ is defined by (20) for $q = 1$. It follows from the proof of Theorem 1 that

$$E_1(p, r) \le e_1(p, r, \widehat{m}) \le \int_T |\psi(t)|\widehat{x}(t)\, d\mu(t) = p\lambda_1\delta^p + r\lambda_2. \quad \square$$

One can easily obtain analogs of Corollaries 1 and 2 for problem (23).

3. The case of homogeneous weight functions

Let T be a cone in a linear space, $T_0 = T$, $|\psi(\cdot)|$ and $|\varphi(\cdot)|$ be homogeneous functions of degrees η, ν, respectively, $\varphi(t) \ne 0$ and $\psi(t) \ne 0$ for almost all $t \in T$, and $\mu(\cdot)$ be a homogeneous measure of degree d. We assume, again, that $1 \le p < q, r < \infty$. For $k \in [0, 1)$ the function $k^{\frac{1}{p-q}}(1-k)^{-\frac{1}{r-q}}$ increases monotonically from 0 to $+\infty$. Consequently, for all $z \in T$ such that $\varphi(z) \ne 0$ and $\psi(z) \ne 0$ (if $p < r$), there exists $k(z)$ for which

$$\frac{k^{\frac{1}{p-q}}(z)}{(1-k(z))^{\frac{1}{r-q}}} = \frac{|\psi(z)|^{\frac{q(p-r)}{(p-q)(r-q)}}}{|\varphi(z)|^{\frac{r}{r-q}}}. \tag{25}$$

Thus, the function $k(z)$ is well defined for almost all $z \in T$.

Theorem 2. *Let $1 \le q < p, r < \infty$, $\varphi(t), \psi(t) \ne 0$ for almost all $t \in T$, and $\nu + d(1/r - 1/p) \ne 0$. Assume that*

$$I_1 = \int_T |\psi(z)|^{\frac{qp}{p-q}} k^{\frac{p}{p-q}}(z)\, d\mu(z) < \infty,$$
$$I_2 = \int_T |\psi(z)|^{\frac{qr}{p-q}} |\varphi(z)|^r k^{\frac{r}{p-q}}(z)\, d\mu(z) < \infty.$$

Then

$$E(p, q, r) = \delta^\gamma I_1^{-\gamma/p} I_2^{-(1-\gamma)/r} (I_1 + I_2)^{1/q},$$

where

$$\gamma = \frac{\nu - \eta - d(1/q - 1/r)}{\nu + d(1/r - 1/p)}, \tag{26}$$

and the method

$$\widehat{m}(y)(t) = k(\xi t)\psi(t)y(t),$$

where

$$\xi = \left(\delta I_1^{-1/p} I_2^{1/r}\right)^{\frac{1}{\nu + d(1/r - 1/p)}}, \tag{27}$$

is optimal recovery method.

Proof. Put

$$\widehat{x}(t) = \left(\frac{q|\psi(t)|^q}{p\lambda_1}\right)^{\frac{1}{p-q}} k^{\frac{1}{p-q}}(\xi t),$$

where $\lambda_1 > 0$ will be specified later. We show that $\widehat{x}(\cdot)$ satisfies (2), where

$$\lambda_2 = r^{-1} q^{\frac{p-r}{p-q}} (p\lambda_1)^{\frac{r-q}{p-q}} \xi^{\nu r - \eta \frac{q(p-r)}{p-q}}. \tag{28}$$

We have

$$p\lambda_1 \widehat{x}^{p-q}(t) = q|\psi(t)|^q k(\xi t),$$

and further,

$$r\lambda_2|\varphi(t)|^r\widehat{x}^{r-q}(t)=r\lambda_2|\varphi(t)|^r\left(\frac{q|\psi(t)|^q}{p\lambda_1}\right)^{\frac{r-q}{p-q}}k^{\frac{r-q}{p-q}}(\xi t).$$

Since $|\varphi(\cdot)|$ and $|\psi(\cdot)|$ are homogeneous it follows by (25) that

$$k^{\frac{r-q}{p-q}}(\xi t)=\frac{|\psi(\xi t)|^{\frac{q(p-r)}{p-q}}}{|\varphi(\xi t)|^r}(1-k(\xi t))=\xi^{\eta\frac{q(p-r)}{p-q}-\nu r}\frac{|\psi(t)|^{\frac{q(p-r)}{p-q}}}{|\varphi(t)|^r}(1-k(\xi t)).$$

Thus,

$$\begin{aligned}r\lambda_2|\varphi(t)|^r\widehat{x}^{r-q}(t)&=r\lambda_2\left(\frac{q}{p\lambda_1}\right)^{\frac{r-q}{p-q}}\xi^{\eta\frac{q(p-r)}{p-q}-\nu r}|\psi(t)|^q(1-k(\xi t))\\&=q|\psi(t)|^q(1-k(\xi t))=q|\psi(t)|^q-p\lambda_1\widehat{x}^{p-q}(t).\end{aligned}$$

Now we show that for

$$\lambda_1=\frac{q}{p}I_1^{\frac{p-q}{p}}\xi^{-\eta q-d\frac{p-q}{p}}\delta^{q-p} \tag{29}$$

the equalities

$$\int_T\widehat{x}^p(t)\,d\mu(t)=\delta^p,\qquad\int_T|\varphi(t)|^r\widehat{x}^r(t)\,d\mu(t)=1$$

hold. In view of the definition of $\widehat{x}(\cdot)$ we need to check that

$$\begin{aligned}\int_T\left(\frac{q|\psi(t)|^q}{p\lambda_1}\right)^{\frac{p}{p-q}}k^{\frac{p}{p-q}}(\xi t)\,d\mu(t)=\delta^p,\\ \int_T|\varphi(t)|^r\left(\frac{q|\psi(t)|^q}{p\lambda_1}\right)^{\frac{r}{p-q}}k^{\frac{r}{p-q}}(\xi t)\,d\mu(t)=1.\end{aligned}$$

Changing $z=\xi t$ and taking into account that functions $|\psi(\cdot)|$, $|\varphi(\cdot)|$ with the measure $\mu(\cdot)$ are homogeneous, we obtain

$$\begin{aligned}&\left(\frac{q}{p\lambda_1}\right)^{\frac{p}{p-q}}I_1=\delta^p\xi^{\frac{\eta qp}{p-q}+d},\\&\left(\frac{q}{p\lambda_1}\right)^{\frac{r}{p-q}}I_2=\xi^{\frac{\eta qr}{p-q}+\nu r+d}.\end{aligned}$$

The validity of these equalities immediately follows from the definitions of λ_1 and ξ.

It follows by Theorem 1, (27)–(29) that

$$\begin{aligned}E^q(p,q,r)&=\frac{p\lambda_1\delta^p+r\lambda_2}{q}=I_1^{\frac{p-q}{p}}\xi^{-\eta q-d\frac{p-q}{p}}\delta^q+\left(\frac{p\lambda_1}{q}\right)^{\frac{r-q}{p-q}}\xi^{\nu r-\eta\frac{q(p-r)}{p-q}}\\&=\delta^{q\gamma}I_1^{-q\gamma/p}I_2^{-q(1-\gamma)/r}(I_1+I_2).\end{aligned}$$

Moreover, the same theorem states that the method

$$\widehat{m}(y)(t)=q^{-1}p\lambda_1\widehat{x}^{p-q}(t)|\psi(t)|^{-q}\psi(t)y(t)=k(\xi t)\psi(t)y(t)$$

is optimal. $\square$

It follows by Theorem 2 and (22) that for all $x(\cdot)\in\mathcal{W}$ such that $\|\varphi(\cdot)x(\cdot)\|_{L_r(T,\mu)}\le 1$ the exact inequality

$$\|\psi(\cdot)x(\cdot)\|_{L_q(T,\mu)}\le C\|x(\cdot)\|^{\gamma}_{L_p(T,\mu)} \tag{30}$$

holds, where

$$C=I_1^{-\gamma/p}I_2^{-(1-\gamma)/r}(I_1+I_2)^{1/q}.$$

(Here and later the exactness means that C cannot be replaced by any other constant smaller than C).

From (30) the following exact inequality can be easily obtained

$$\|\psi(\cdot)x(\cdot)\|_{L_q(T,\mu)}\le C\|x(\cdot)\|^{\gamma}_{L_p(T,\mu)}\|\varphi(\cdot)x(\cdot)\|^{1-\gamma}_{L_r(T,\mu)}, \tag{31}$$

which holds for all $x(\cdot)\in\mathcal{W}$, $x(\cdot)\ne 0$.

Let $|w(\cdot)|$, $|w_0(\cdot)|$, and $|w_1(\cdot)|$ be homogeneous functions of degrees θ, θ_0, and θ_1, respectively. We assume that $w(t),w_0(t),w_1(t)\ne 0$ for almost all $t\in T$ and $1\le q<p,r<\infty$. Then for almost all $z\in T$ such that $w(z),w_0(z),w_1(z)\ne 0$ there exists $\widetilde{k}(z)$ satisfying

$$\frac{\widetilde{k}^{\frac{1}{p-q}}(z)}{(1-\widetilde{k}(z))^{\frac{1}{r-q}}}=\left|\frac{w(z)}{w_1(z)}\right|^{\frac{r}{r-q}}\left|\frac{w_0(z)}{w(z)}\right|^{\frac{p}{p-q}}.$$

Put

$$\widetilde{\theta}=\theta+d/q,\qquad \widetilde{\theta}_0=\theta_0+d/p,\qquad \widetilde{\theta}_1=\theta_1+d/r. \tag{32}$$

Corollary 3. *Let $1\le q<p,r<\infty$, $w(t),w_0(t),w_1(t)\ne 0$ for almost all $t\in T$, and $\widetilde{\theta}_0\ne\widetilde{\theta}_1$. Assume that*

$$\widetilde{I}_1=\int_T\left|\frac{w(z)}{w_0(z)}\right|^{\frac{qp}{p-q}}\widetilde{k}^{\frac{p}{p-q}}(z)\,d\mu(z)<\infty,$$
$$\widetilde{I}_2=\int_T\frac{|w(z)|^{\frac{qr}{p-q}}}{|w_0(z)|^{\frac{pr}{p-q}}}|w_1(z)|^r\widetilde{k}^{\frac{r}{p-q}}(z)\,d\mu(z)<\infty.$$

Then for all $x(\cdot)\ne 0$ such that $w_0(\cdot)x(\cdot)\in L_p(T,\mu)$ and $w_1(\cdot)x(\cdot)\in L_r(T,\mu)$ the exact inequality

$$\|w(\cdot)x(\cdot)\|_{L_q(T,\mu)}\le\widetilde{C}\|w_0(\cdot)x(\cdot)\|^{\widetilde{\gamma}}_{L_p(T,\mu)}\|w_1(\cdot)x(\cdot)\|^{1-\widetilde{\gamma}}_{L_r(T,\mu)} \tag{33}$$

holds; here

$$\widetilde{C}=\widetilde{I}_1^{-\widetilde{\gamma}/p}\widetilde{I}_2^{-(1-\widetilde{\gamma})/r}(\widetilde{I}_1+\widetilde{I}_2)^{1/q},\qquad \widetilde{\gamma}=\frac{\widetilde{\theta}_1-\widetilde{\theta}}{\widetilde{\theta}_1-\widetilde{\theta}_0}.$$

Proof. Put

$$\psi(x)=\frac{w(x)}{w_0(x)},\qquad \varphi(x)=\frac{w_1(x)}{w_0(x)}.$$

Then $|\psi(\cdot)|$ and $|\varphi(\cdot)|$ are homogeneous functions of degrees $\eta=\theta-\theta_0$ and $\nu=\theta_1-\theta_0$, respectively. It follows by (31) that for all $x(\cdot)\in\mathcal{W}$, $x(\cdot)\ne 0$, the exact inequality

$$\|\psi(\cdot)x(\cdot)\|_{L_q(T,\mu)}\le\widetilde{C}\|x(\cdot)\|^{\widetilde{\gamma}}_{L_p(T,\mu)}\|\varphi(\cdot)x(\cdot)\|^{1-\widetilde{\gamma}}_{L_r(T,\mu)}$$

holds. Substituting $x(\cdot)=w_0(\cdot)y(\cdot)$, we obtain (33). □

The well-known Carlson inequality [4]

$$\|x(t)\|_{L_1(\mathbb{R}_+)} \le \sqrt{\pi}\|x(t)\|_{L_2(\mathbb{R}_+)}^{1/2}\|tx(t)\|_{L_2(\mathbb{R}_+)}^{1/2} \tag{34}$$

was generalized in many directions (see [5,1,3]). Inequality (33) is also a generalization of the Carlson inequality.

Let $1 \le p < q, r < \infty$, T be a cone in $\mathbb{R}^d$, $d\mu(t) = dt$, $|\psi(\cdot)|$ and $|\varphi(\cdot)|$ be homogeneous functions of degrees η, ν, respectively, $\varphi(t) \ne 0$ and $\psi(t) \ne 0$ for almost all $t \in T$. Thus $\mu(\cdot)$ is a homogeneous measure of degree d. Consider the polar transformation

$$\begin{aligned} x_1 &= \rho\cos\omega_1,\\ x_2 &= \rho\sin\omega_1\cos\omega_2,\\ &\dots\dots\dots\dots\dots\dots\dots\dots\\ x_{d-1} &= \rho\sin\omega_1\sin\omega_2\dots\sin\omega_{d-2}\cos\omega_{d-1},\\ x_d &= \rho\sin\omega_1\sin\omega_2\dots\sin\omega_{d-2}\sin\omega_{d-1}. \end{aligned}$$

Set $\omega = (\omega_1, \dots, \omega_{d-1})$,

$$\begin{aligned} \widetilde{\psi}(\omega) &= \rho^{-\eta}|\psi(\rho\cos\omega_1, \dots, \rho\sin\omega_1\sin\omega_2\dots\sin\omega_{d-2}\sin\omega_{d-1})|,\\ \widetilde{\varphi}(\omega) &= \rho^{-\nu}|\varphi(\rho\cos\omega_1, \dots, \rho\sin\omega_1\sin\omega_2\dots\sin\omega_{d-2}\sin\omega_{d-1})|. \end{aligned} \tag{35}$$

Denote by Ω the range of ω. Since T is a cone, Ω does not depend on ρ. Put

$$J(\omega) = \sin^{d-2}\omega_1\sin^{d-3}\omega_2\dots\sin\omega_{d-2}.$$

By (25) we obtain the following equality for $k(\cdot)$:

$$\frac{k^{\frac{1}{p-q}}(\rho,\omega)}{(1-k(\rho,\omega))^{\frac{1}{r-q}}} = \rho^{\frac{\eta q(p-r)-\nu r(p-q)}{(p-q)(r-q)}}\frac{\widetilde{\psi}^{\frac{q(p-r)}{(p-q)(r-q)}}(\omega)}{\widetilde{\varphi}^{\frac{r}{r-q}}(\omega)}. \tag{36}$$

Assume that $\gamma \in (0, 1)$, where γ is defined by (26). Put

$$\frac{1}{q^*} = \frac{1}{q} - \frac{\gamma}{p} - \frac{1-\gamma}{r}. \tag{37}$$

It is easy to verify that $q^* > q \ge 1$. Moreover,

$$q^* = \frac{pqr(\nu + d(1/r - 1/p))}{\nu r(p-q) - \eta q(p-r)}.$$

Theorem 3. *Let $1 \le q < p, r < \infty$, $\gamma \in (0, 1)$, and $\widetilde{\varphi}(\omega), \widetilde{\psi}(\omega) \ne 0$ for almost all $\omega \in \Omega$. Assume that*

$$I = \int_\Omega \frac{\widetilde{\psi}^{q^*}(\omega)}{\widetilde{\varphi}^{q^*(1-\gamma)}(\omega)}J(\omega)\,d\omega < \infty.$$

Then

$$E(p, q, r) = C_1\delta^\gamma,$$

where

$$C_1 = \gamma^{-\frac{\gamma}{p}}(1-\gamma)^{-\frac{1-\gamma}{r}}\left(\frac{B(q^*\gamma/p, q^*(1-\gamma)/r)\,I}{|\nu + d(1/r-1/p)|(\gamma r + (1-\gamma)p)}\right)^{1/q^*},$$

where $B(\cdot,\cdot)$ is the beta-function. Moreover, the method

$$\widehat{m}(y)(t) = k\left(\xi_1^{\frac{1}{\nu+d(1/r-1/p)}}t\right)\psi(t)y(t),$$

where

$$\xi_1=\delta\left(\gamma^{q-r}(1-\gamma)^{p-q}C_1^{p-r}\right)^{\frac{q^*}{pqr}},$$

is optimal recovery method.

Proof. Using Theorem 2, we obtain

$$\begin{aligned}I_1&=\int_T|\psi(z)|^{\frac{qp}{p-q}}k^{\frac{p}{p-q}}(z)\,dz\\&=\int_\Omega\widetilde{\psi}^{\frac{qp}{p-q}}(\omega)J(\omega)\,d\omega\int_0^{+\infty}\rho^{\frac{\eta qp}{p-q}+d-1}k^{\frac{p}{p-q}}(\rho,\omega)\,d\rho.\end{aligned}$$

By (36) we have

$$\rho^{\nu r(p-q)-\eta q(p-r)}=\frac{(1-k(\rho,\omega))^{p-q}}{k^{r-q}(\rho,\omega)}\frac{\widetilde{\psi}^{q(p-r)}(\omega)}{\widetilde{\varphi}^{r(p-q)}(\omega)}. \tag{38}$$

Fixing ω, we pass to k

$$\begin{aligned}d\rho^{\frac{\eta qp}{p-q}+d}&=\left(\frac{\widetilde{\psi}^{q(p-r)}(\omega)}{\widetilde{\varphi}^{r(p-q)}(\omega)}\right)^{\zeta}d\frac{(1-k)^{(p-q)\zeta}}{k^{(r-q)\zeta}}\\&=-\zeta\left(\frac{\widetilde{\psi}^{q(p-r)}(\omega)}{\widetilde{\varphi}^{r(p-q)}(\omega)}\right)^{\zeta}\frac{(1-k)^{(p-q)\zeta-1}}{k^{(r-q)\zeta+1}}(r-q+(p-r)k)\,dk,\end{aligned}$$

where

$$\zeta=\frac{\eta qp+d(p-q)}{(p-q)(\nu r(p-q)-\eta q(p-r))}=\frac{q^*(1-\gamma)}{r(p-q)}.$$

Consequently,

$$\begin{aligned}&\int_0^{+\infty}\rho^{\frac{\eta qp}{p-q}+d-1}k^{\frac{p}{p-q}}(\rho,\omega)\,d\rho\\&\quad=\frac{p-q}{\eta qp+d(p-q)}\int_0^{+\infty}k^{\frac{p}{p-q}}(\rho,\omega)\,d\rho^{\frac{\eta qp}{p-q}+d}\\&\quad=\frac{1}{|\nu r(p-q)-\eta q(p-r)|}\left(\frac{\widetilde{\psi}^{q(p-r)}(\omega)}{\widetilde{\varphi}^{r(p-q)}(\omega)}\right)^{\zeta}(K_1+K_2),\end{aligned}$$

where

$$\begin{aligned}K_1&=(r-q)\int_0^1k^{\widehat{p}}(1-k)^{\widehat{q}-1}\,dk=(r-q)B(\widehat{p}+1,\widehat{q}),\\K_2&=(p-r)\int_0^1k^{\widehat{p}+1}(1-k)^{\widehat{q}-1}\,dk=(p-r)B(\widehat{p}+2,\widehat{q})\\&=(p-r)\frac{\widehat{p}+1}{\widehat{p}+\widehat{q}+1}B(\widehat{p}+1,\widehat{q}),\\\widehat{p}&=\frac{qr(\nu-\eta)-d(r-q)}{\nu r(p-q)-\eta q(p-r)}=q^*\frac{\gamma}{p},\qquad\widehat{q}=\frac{\eta qp+d(p-q)}{\nu r(p-q)-\eta q(p-r)}=q^*\frac{1-\gamma}{r}.\end{aligned}$$

Thus,

$$\begin{aligned}K_1+K_2&=p\frac{\nu r(p-q)-\eta q(p-r)}{\nu pr+d(p-r)}B(\widehat{p}+1,\widehat{q})=\frac{pq}{q^*}B(\widehat{p}+1,\widehat{q})\\&=\frac{q\gamma}{q^*}\left(\frac{\gamma}{p}+\frac{1-\gamma}{r}\right)^{-1}B(\widehat{p},\widehat{q}).\end{aligned}$$

The analogous calculations give

$$\begin{aligned} I_2 &= \int_T |\psi(z)|^{\frac{qr}{p-q}} |\varphi(z)|^r k^{\frac{r}{p-q}}(z)\, d\mu(z) \\ &= \int_\Omega \widetilde{\psi}^{\frac{qr}{p-q}}(\omega)\widetilde{\varphi}^r(\omega) J(\omega)\, d\omega \int_0^{+\infty} \rho^{\frac{\eta qr}{p-q}+\nu r+d-1} k^{\frac{r}{p-q}}(\rho,\omega)\, d\rho. \end{aligned}$$

Fixing ω, we pass to k

$$\begin{aligned} d\rho^{\frac{\eta qr}{p-q}+\nu r+d} &= \left(\frac{\widetilde{\psi}^{q(p-r)}(\omega)}{\widetilde{\varphi}^{r(p-q)}(\omega)}\right)^{\zeta_1} d\frac{(1-k)^{(p-q)\zeta_1}}{k^{(r-q)\zeta_1}} \\ &= -\zeta_1 \left(\frac{\widetilde{\psi}^{q(p-r)}(\omega)}{\widetilde{\varphi}^{r(p-q)}(\omega)}\right)^{\zeta_1} \frac{(1-k)^{(p-q)\zeta_1-1}}{k^{(r-q)\zeta_1+1}}(r-q+(p-r)k)\, dk, \end{aligned}$$

where

$$\zeta_1 = \frac{\eta qr + (\nu r + d)(p-q)}{(p-q)(\nu r(p-q) - \eta q(p-r))} = \frac{q^*(1-\gamma)}{r(p-q)} + \frac{1}{p-q}.$$

We have

$$\begin{aligned} &\int_0^{+\infty} \rho^{\frac{\eta qr}{p-q}+\nu r+d-1} k^{\frac{r}{p-q}}(\rho,\omega)\, d\rho \\ &\quad = \frac{p-q}{\eta qr + (\nu r + d)(p-q)} \int_0^{+\infty} k^{\frac{r}{p-q}}(\rho,\omega)\, d\rho^{\frac{\eta qr}{p-q}+\nu r+d} \\ &\quad = \frac{1}{|\nu r(p-q) - \eta q(p-r)|} \left(\frac{\widetilde{\psi}^{q(p-r)}(\omega)}{\widetilde{\varphi}^{r(p-q)}(\omega)}\right)^{\zeta_1} (L_1 + L_2), \end{aligned}$$

where

$$\begin{aligned} L_1 &= (r-q)\int_0^1 k^{\widehat{p}-1}(1-k)^{\widehat{q}}\, dk = (r-q)B(\widehat{p}, \widehat{q}+1), \\ L_2 &= (p-r)\int_0^1 k^{\widehat{p}}(1-k)^{\widehat{q}}\, dk = (p-r)B(\widehat{p}+1, \widehat{q}+1) \\ &= (p-r)\frac{\widehat{p}}{\widehat{p}+\widehat{q}+1} B(\widehat{p}, \widehat{q}+1). \end{aligned}$$

Thus,

$$\begin{aligned} L_1 + L_2 &= r\frac{\nu r(p-q) - \eta q(p-r)}{\nu pr + d(p-r)} B(\widehat{p}, \widehat{q}+1) = \frac{qr}{q^*} B(\widehat{p}, \widehat{q}+1) \\ &= \frac{q(1-\gamma)}{q^*} \left(\frac{\gamma}{p} + \frac{1-\gamma}{r}\right)^{-1} B(\widehat{p}, \widehat{q}). \end{aligned}$$

We obtain

$$\begin{aligned} I_1 &= \frac{\gamma}{pr|\nu + d(1/r - 1/p)|} \left(\frac{\gamma}{p} + \frac{1-\gamma}{r}\right)^{-1} B(\widehat{p}, \widehat{q}) I, \\ I_2 &= \frac{1-\gamma}{pr|\nu + d(1/r - 1/p)|} \left(\frac{\gamma}{p} + \frac{1-\gamma}{r}\right)^{-1} B(\widehat{p}, \widehat{q}) I. \end{aligned}$$

It remains to apply Theorem 2. □

Note that for $d = 1$ we have $I = 1$ when $T = \mathbb{R}_+$ and $I = 2$ when $T = \mathbb{R}$.

Assume that $|w(\cdot)|$, $|w_0(\cdot)|$, and $|w_1(\cdot)|$ are homogeneous functions of degrees θ, θ_0, and θ_1, respectively. Define $\widetilde{w}(\cdot)$, $\widetilde{w}_0(\cdot)$, $\widetilde{w}_1(\cdot)$ by the analogy with (35).

From Theorem 2 (analogously to Corollary 3) we immediately obtain

Corollary 4 *([3][1]). Suppose that $w(t), w_0(t), w_1(t) \neq 0$ for almost all $t \in T$, $1 \le q < p, r < \infty$, $\widetilde{\gamma} \in (0,1)$, where*

$$\widetilde{\gamma} = \frac{\widetilde{\theta}_1 - \widetilde{\theta}}{\widetilde{\theta}_1 - \widetilde{\theta}_0},$$

and $\widetilde{\theta}$, $\widetilde{\theta}_0$, and $\widetilde{\theta}_1$ are defined by (32). Moreover, assume that

$$\widetilde{I} = \int_{\Omega} \frac{\widetilde{w}^{\widetilde{q}}(\omega)}{\widetilde{w}_0^{\widetilde{q}\widetilde{\gamma}}(\omega)\widetilde{w}_1^{\widetilde{q}(1-\widetilde{\gamma})}(\omega)} J(\omega)\, d\omega < \infty,$$

where

$$\frac{1}{\widetilde{q}} = \frac{1}{q} - \frac{\widetilde{\gamma}}{p} - \frac{1-\widetilde{\gamma}}{r}.$$

Then the exact inequality

$$\|w(\cdot)x(\cdot)\|_{L_q(T,\mu)} \le \widetilde{C}_1 \|w_0(\cdot)x(\cdot)\|_{L_p(T,\mu)}^{\widetilde{\gamma}} \|w_1(\cdot)x(\cdot)\|_{L_r(T,\mu)}^{1-\widetilde{\gamma}} \tag{39}$$

holds; here

$$\widetilde{C}_1 = \widetilde{\gamma}^{-\frac{\widetilde{\gamma}}{p}}(1-\widetilde{\gamma})^{-\frac{1-\widetilde{\gamma}}{r}} \left(\frac{B\left(\widetilde{q}\widetilde{\gamma}/p, \widetilde{q}(1-\widetilde{\gamma})/r\right)\widetilde{I}}{|\theta_1 - \theta_0|(\widetilde{\gamma}r + (1-\widetilde{\gamma})p)} \right)^{1/\widetilde{q}}.$$

Put

$$w_0(t) = 1, \qquad w_1(t) = t^{1-(\lambda+1)/p}, \qquad w_2(t) = t^{1+(\mu-1)/q}.$$

From Corollary 4 we obtain

Corollary 5 *([5]). Let $1 < p, q < \infty$ and $\lambda, \mu > 0$. Put*

$$\alpha = \frac{\mu}{p\mu + q\lambda}, \qquad \beta = \frac{\lambda}{p\mu + q\lambda}.$$

Then the exact inequality

$$\|x(t)\|_{L_1(\mathbb{R}_+)} \le C \|t^{1-(\lambda+1)/p} x(t)\|_{L_p(\mathbb{R}_+)}^{p\alpha} \|t^{1+(\mu-1)/q} x(t)\|_{L_q(\mathbb{R}_+)}^{q\beta}$$

holds; here

$$C = \frac{1}{(p\alpha)^{\alpha}(q\beta)^{\beta}} \left(\frac{1}{\lambda+\mu} B\left(\frac{\alpha}{1-\alpha-\beta}, \frac{\beta}{1-\alpha-\beta} \right) \right)^{1-\alpha-\beta}.$$

Using Theorem 1′ and calculations from the proofs of Theorems 2 and 3 we obtain

[1] The exact constant in [3] (formula (10)) was given with a misprint.

Theorem 3′. *Let* $1 < p, r < \infty$, $\widetilde{\varphi}(\omega)$, $\widetilde{\psi}(\omega) \neq 0$ *for almost all* $\omega \in \Omega$ *and* γ, q^*, I, $k(\cdot)$, C_1, ξ_1 *as above but for* $q = 1$. *Assume that* $\gamma \in (0,1)$ *and* $I < \infty$. *Then*

$$E_1(p,r) = C_1\delta^{\gamma}.$$

Moreover, the method

$$\widehat{m}(y) = \int_T k\left(\xi_1^{\frac{1}{\nu+d(1/r-1/p)}} t\right)\psi(t)y(t)\,d\mu(t)$$

is optimal recovery method.

4. Optimal recovery of functions from a noisy Fourier transform

Let S be the Schwartz space of rapidly decreasing C^{∞}-functions on $\mathbb{R}$, S' the corresponding space of distributions, and let $F\colon S' \to S'$ be the Fourier transform. We let $\mathcal{F}_p$ denote the space of distribution $x(\cdot)$ in S' for which

$$\|x(\cdot)\|_p = \left(\int_{\mathbb{R}} |Fx(t)|^p\,dt\right)^{1/p} < \infty, \quad 1 \le p < \infty.$$

We set

$$\mathcal{F}_p^n = \{x(\cdot) \in S' : \|x^{(n)}(\cdot)\|_p < \infty\},$$
$$F_p^n = \{x(\cdot) \in \mathcal{F}_p^n : \|x^{(n)}(\cdot)\|_p \le 1\}.$$

Assume that the Fourier transform of a function $x(\cdot) \in F_r^n \cap \mathcal{F}_p$ is known on $\mathbb{R}$ to within $\delta > 0$ in the metric of $L_p(\mathbb{R})$. In other words, we know a function $y(\cdot) \in L_p(\mathbb{R})$ such that $\|Fx(\cdot) - y(\cdot)\|_{L_p(\mathbb{R})} \le \delta$. How should we best use this information to recover the lth derivative of the function in the metric $\mathcal{F}_q$, $0 \le l < n$? By recovery methods here we mean all possible mappings $m\colon L_p(\mathbb{R}) \to \mathcal{F}_q$. The error of a method is, by definition, the quantity

$$e_{p,q,r}(m) = \sup_{\substack{x(\cdot)\in F_r^n\cap\mathcal{F}_p,\ y(\cdot)\in L_p(\mathbb{R})\\ \|Fx(\cdot)-y(\cdot)\|_{L_p(\Delta_\sigma)}\le\delta}} \|x^{(l)}(\cdot) - m(y)(\cdot)\|_q.$$

The optimal recovery error is defined as follows:

$$E_{p,q,r} = \inf_{m\colon L_p(\mathbb{R})\to\mathcal{F}_q} e_{p,q,r}(m).$$

A method on which this lower bound is attained is called optimal.

It is readily checked that this problem is a special case of the general problem (1) with $T = T_0 = \mathbb{R}$, $\psi(t) = (it)^l$, $\varphi(t) = (it)^n$.

The cases (1) $1 \le q = r < p < \infty$, (2) $1 \le q = p < r < \infty$, (3) $1 \le q = p = r < \infty$, and (4) $1 \le q < p = r < \infty$ were studied in [14].

For the case $1 \le q < p, r < \infty$ we can apply Theorem 3. In this case

$$\frac{k^{\frac{1}{p-q}}(t)}{(1-k(t))^{\frac{1}{r-q}}} = |t|^{\frac{lq(p-r)-nr(p-q)}{(p-q)(r-q)}}, \qquad \gamma = \frac{n-l-1/q+1/r}{n+1/r-1/p},$$

and $I = 2$. It is easy to verify that if $n > l + 1/q - 1/r$, then $\gamma \in (0,1)$. Thus, it follows by Theorem 3.

Theorem 4. *Let* $1 \le q < p, r < \infty$ *and* $n > l + 1/q - 1/r$. *Then*

$$E_{p,q,r} = C_1\delta^{\gamma}, \tag{40}$$

where

$$C_1 = \gamma^{-\frac{\gamma}{p}}(1-\gamma)^{-\frac{1-\gamma}{r}}\left(\frac{2B\left(q^*\gamma/p, q^*(1-\gamma)/r\right)}{(n+1/r-1/p)(\gamma r+(1-\gamma)p)}\right)^{1/q^*}$$

and q^* is defined by (37). Moreover, the method $\widehat{m}(y)(\cdot) = F^{-1}Y_y(\cdot)$ is optimal, where

$$Y_y(t) = (it)^l k\left(\xi_1^{\frac{1}{n+1/r-1/p}} t\right) y(t), \qquad \xi_1 = \delta\left(\gamma^{q-r}(1-\gamma)^{p-q} C_1^{p-r}\right)^{\frac{q^*}{pqr}}.$$

Note that case (4) immediately follows from Theorem 4 for $p = r$. In cases (1)–(3) the optimal recovery error coincides with the limits $\lim_{r\to q} E_{p,q,r}$, $\lim_{p\to q} E_{p,q,r}$, $\lim_{p\to q} E_{p,q,p}$, respectively, where $E_{p,q,r}$ is given by (40).

5. Optimal recovery of derivatives and generalized Carlson–Levin–Taikov inequalities

For functions $x(\cdot) \in L_2(\mathbb{R})$ whose $(n-1)$st derivative is locally absolutely continuous and $0 \le k \le n-1$, L. V. Taikov [16] obtained exact inequality

$$|x^{(k)}(0)| \le K\|x(\cdot)\|_{L_2(\mathbb{R})}^{\frac{2n-2k-1}{2n}} \|x^{(n)}(\cdot)\|_{L_2(\mathbb{R})}^{\frac{2k+1}{2n}},$$

where

$$K = \left(\frac{2k+1}{2n-2k-1}\right)^{\frac{2n-2k-1}{4n}} \left((2k+1)\sin\frac{2k+1}{2n}\pi\right)^{-1/2}.$$

Passing to the Fourier transform we have the following equivalent inequality

$$\left|\frac{1}{2\pi}\int_{\mathbb{R}} t^k Fx(t)\,dt\right| \le K\left(\frac{1}{2\pi}\int_{\mathbb{R}} |Fx(t)|^2\,dt\right)^{\frac{2n-2k-1}{4n}} \times \left(\frac{1}{2\pi}\int_{\mathbb{R}} t^{2n}|Fx(t)|^2\,dt\right)^{\frac{2k+1}{4n}}.$$

Set $g(t) = t^k Fx(t)$. Then we obtain the following inequality

$$\left|\int_{\mathbb{R}} g(t)\,dt\right| \le K\sqrt{2\pi}\left(\int_{\mathbb{R}} t^{-2k}|g(t)|^2\,dt\right)^{\frac{2n-2k-1}{4n}} \times \left(\int_{\mathbb{R}} t^{2(n-k)}|g(t)|^2\,dt\right)^{\frac{2k+1}{4n}}.$$

Put $p = q = 2$, $\lambda = 2k+1$, and $\mu = 2n-2k-1$. Then by Corollary 4 we have

$$\int_0^\infty |g(t)|\,dt \le C\left(\int_0^\infty t^{-2k}|g(t)|^2\,dt\right)^{\frac{2n-2k-1}{4n}} \times \left(\int_0^\infty t^{2(n-k)}|g(t)|^2\,dt\right)^{\frac{2k+1}{4n}},$$

where

$$C = \left(\frac{2k+1}{2n-2k-1}\right)^{\frac{2n-2k-1}{4n}} (2k+1)^{-1/2} B^{1/2}\left(\frac{2n-2k-1}{2n}, \frac{2k+1}{2n}\right).$$

Since

$$B\left(1-\frac{2k+1}{2n}, \frac{2k+1}{2n}\right) = \frac{\pi}{\sin\frac{2k+1}{2n}\pi}$$

we have

$$C = \sqrt{\pi}\left(\frac{2k+1}{2n-2k-1}\right)^{\frac{2n-2k-1}{4n}} \left((2k+1)\sin\frac{2k+1}{2n}\pi\right)^{-1/2}.$$

From the inequality

$$a_1 b_1 + a_2 b_2 \le 2^{1-s-t}(a_1^{1/r} + a_2^{1/r})^r (b_1^{1/s} + b_2^{1/s})^s$$

it follows that

$$\begin{aligned}\int_{\mathbb{R}}|g(t)|\,dt &= \int_{-\infty}^{0}|g(t)|\,dt+\int_{0}^{\infty}|g(t)|\,dt\\ &\le C\left(\int_{-\infty}^{0}t^{-2k}|g(t)|^2\,dt\right)^{\frac{2n-2k-1}{4n}}\left(\int_{-\infty}^{0}t^{2(n-k)}|g(t)|^2\,dt\right)^{\frac{2k+1}{4n}}\\ &\quad+C\left(\int_{0}^{\infty}t^{-2k}|g(t)|^2\,dt\right)^{\frac{2n-2k-1}{4n}}\left(\int_{0}^{\infty}t^{2(n-k)}|g(t)|^2\,dt\right)^{\frac{2k+1}{4n}}\\ &\le \sqrt{2}C\left(\int_{\mathbb{R}}t^{-2k}|g(t)|^2\,dt\right)^{\frac{2n-2k-1}{4n}}\left(\int_{\mathbb{R}}t^{2(n-k)}|g(t)|^2\,dt\right)^{\frac{2k+1}{4n}}.\end{aligned}$$

Thus Taikov's inequality follows from Levin's inequality.

This inequality is closely connected with the problem of optimal recovery of derivatives from inaccurate information about the Fourier transform (see [10]). We consider such problem in multidimensional case.

Consider linear operators $D_1\colon L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)\cap C(\mathbb{R}^d)$ and $D_2\colon L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)$ (D_1 and D_2 are not necessary differentiation operators). Put

$$W=\{x(\cdot)\in L_2(\mathbb{R}^d):\|D_2x(\cdot)\|_{L_2(\mathbb{R}^d)}\le 1\}.$$

We consider the problem of optimal recovery of $D_1x(\tau)$, $\tau\in\mathbb{R}^d$, on the class W from the information about $x(\cdot)$, given inaccurately in $L_2(\mathbb{R}^d)$-metric.

As recovery methods we consider all possible mappings $m\colon L_2(\mathbb{R}^d)\to\mathbb{C}$ or $\mathbb{R}$. The error of a method m is defined as

$$e(m)=\sup_{\substack{x(\cdot)\in W,\ y(\cdot)\in L_2(\mathbb{R}^d)\\ \|x(\cdot)-y(\cdot)\|_{L_2(\mathbb{R}^d)}\le\delta}}|D_1x(\tau)-m(y)|.$$

The quantity

$$E=\inf_{m\colon L_2(\mathbb{R}^d)\to\mathbb{C}(\mathbb{R})}e(m)\tag{41}$$

is known as the optimal recovery error, and a method on which this infimum is attained is called optimal.

For the case when $d=1$, $D_1x(\cdot)=x^{(k)}(\cdot)$, and $D_2x(\cdot)=x^{(n)}(\cdot)$, $0\le k<n$, similar problems were considered in [10].

Let $d_1(t)$ and $d_2(\cdot)$ be measurable functions on R^d. Put

$$X=\{x(\cdot)\in L_2(\mathbb{R}^d):d_2(\cdot)Fx(\cdot)\in L_2(\mathbb{R}^d)\}.$$

We define the operator D_2 as follows

$$D_2x(\cdot)=F^{-1}(d_2(\cdot)Fx(\cdot))(\cdot).$$

Assume that $d_1(\cdot)Fx(\cdot)\in L_2(\mathbb{R}^d)$ for all $x(\cdot)\in X$ and the operator D_1 which is defined by the equality

$$D_1x(\cdot)=F^{-1}(d_1(\cdot)Fx(\cdot))(\cdot)$$

maps X to $L_2(\mathbb{R}^d)\cap C(\mathbb{R}^d)$.

Let $|d_1(\cdot)|$ and $|d_2(\cdot)|$ be homogeneous functions of degrees k, n, respectively (k and n are not necessarily integer), $d_j(t)\ne 0$, $j=1,2$, for almost all $t\in\mathbb{R}^d$. Put

$$\widetilde{d}_1(\omega)=\rho^{-k}|d_1(\rho\cos\omega_1,\dots,\rho\sin\omega_1\sin\omega_2\dots\sin\omega_{d-2}\sin\omega_{d-1})|,$$
$$\widetilde{d}_2(\omega)=\rho^{-n}|d_2(\rho\cos\omega_1,\dots,\rho\sin\omega_1\sin\omega_2\dots\sin\omega_{d-2}\sin\omega_{d-1})|.$$

By Plancherel's theorem we have

$$W=\left\{x(\cdot)\in L_2(\mathbb{R}^d):\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|d_2(t)Fx(t)|^2\,dt\le 1\right\},$$

$$\|x(\cdot)-y(\cdot)\|_{L_2(\mathbb{R}^d)}=\frac{1}{(2\pi)^{d/2}}\|Fx(\cdot)-Fy(\cdot)\|_{L_2(\mathbb{R}^d)}.$$

Moreover,

$$D_1x(\tau)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}d_1(t)Fx(t)e^{i\langle\tau,t\rangle}\,dt,$$

where $\langle\tau,t\rangle=\tau_1t_1+\cdots+\tau_dt_d$. Thus we obtain problem (23) with $p=r=2$, $\delta_1=\delta(2\pi)^{d/2}$,

$$\psi(t)=\frac{1}{(2\pi)^d}d_1(t)e^{i\langle\tau,t\rangle},\qquad\varphi(t)=\frac{1}{(2\pi)^{d/2}}d_2(t).$$

By Theorem 3′ we have

Theorem 5. *Let $k\ge 0$ and $n>k+d/2$. Assume that*

$$I=\int_{\Pi_{d-1}}\frac{\widetilde{d}_1^{\,2}(\omega)}{\widetilde{d}_2^{\,\frac{2k+d}{n}}(\omega)}J(\omega)\,d\omega<\infty,\qquad\Pi_{d-1}=[0,\pi]^{d-2}\times[0,2\pi].$$

Then

$$E=\frac{(\pi I)^{1/2}}{(2\pi)^{d/2}}K_d(k,n)\delta^{\frac{2n-2k-d}{2n}},$$

where

$$K_d(k,n)=\left(\frac{2k+d}{2n-2k-d}\right)^{\frac{2n-2k-d}{4n}}\left((2k+d)\sin\frac{2k+d}{2n}\pi\right)^{-1/2}.$$

Moreover, the method

$$\widehat{m}(y)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}d_1(t)\left(1+\frac{\delta^2(2k+d)}{(2\pi)^d(2n-2k-d)}\right)^{-1}y(t)e^{i\langle\tau,t\rangle}\,dt$$

is optimal recovery method.

By this theorem analogous to (31) we obtain the exact inequality

$$|D_1x(\tau)|\le\frac{(\pi I)^{1/2}}{(2\pi)^{d/2}}K_d(k,n)\|x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\frac{2n-2k-d}{2n}}\|D_2x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\frac{2k+d}{2n}}$$

or

$$\|D_1x(\cdot)\|_{L_\infty(\mathbb{R}^d)}\le\frac{(\pi I)^{1/2}}{(2\pi)^{d/2}}K_d(k,n)\|x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\frac{2n-2k-d}{2n}}\|D_2x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\frac{2k+d}{2n}}.\tag{42}$$

Now we consider some examples. Define the operator $(-\Delta)^{n/2}$, $n\ge 0$, as follows

$$(-\Delta)^{n/2}x(\cdot)=F^{-1}(|t|^nFx(t))(\cdot).$$

Put $d_1(t)=|t|^k$ and $d_2(t)=|t|^n$. Then problem (41) is the problem of optimal recovery of $(-\Delta)^{k/2}x(\tau)$ on the class

$$W=\{x(\cdot)\in L_2(\mathbb{R}^d):\|(-\Delta)^{n/2}x(\cdot)\|_{L_2(\mathbb{R}^d)}\le 1\}$$

by the inaccurate information about $x(\cdot)$.

By Theorem 5 we obtain

Corollary 6. *Let $n > k + d/2$. Then*

$$E = C_d(k,n)\delta^{\frac{2n-2k-d}{2n}}, \qquad C_d(k,n) = \frac{K_d(k,n)}{(2^{d-1}\pi^{d/2-1}\Gamma(d/2))^{1/2}},$$

and the method

$$\widehat{m}(y) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |t|^k \left(1 + \frac{\delta^2(2k+d)}{(2\pi)^d(2n-2k-d)}\right)^{-1} y(t)e^{i\langle\tau,t\rangle}\,dt$$

is optimal.

By (42) we get the exact inequality

$$\|(-\Delta)^{k/2}x(\cdot)\|_{L_\infty(\mathbb{R}^d)} \le C_d(k,n)\|x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\frac{2n-2k-d}{2n}} \|(-\Delta)^{n/2}x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\frac{2k+d}{2n}}.$$

Consider one more example. Let $\alpha = (\alpha_1,\dots,\alpha_d) \in \mathbb{Z}_+^d$. We define D^α (the derivative of order α) as follows:

$$D^\alpha x(\cdot) = F^{-1}((it)^\alpha Fx(t))(\cdot),$$

where $(it)^\alpha = (it_1)^{\alpha_1}\cdots(it_d)^{\alpha_d}$. Let $D_1 = D^\alpha$ and $D_2 = (-\Delta)^{n/2}$. Then (41) is the problem of optimal recovery of $D^\alpha x(\tau)$ on the class W by the inaccurate information about $x(\cdot)$.

From the well-known Dirichlet formula we have

$$\int_{\substack{x_1\ge0,\dots,x_d\ge0\\ x_1^2+\dots+x_d^2\le1}} x_1^{p_1-1}\dots x_d^{p_d-1}\,dx_1\dots dx_d = \frac{\Gamma(p_1/2)\dots\Gamma(p_d/2)}{2^d\Gamma(p_1/2+\dots+p_d/2+1)},$$

$p_1,\dots,p_d > 0$. Using this formula and passing to the polar transformation we obtain

$$I(p_1,\dots,p_d) = \int_{\Pi_{d-1}} \Phi(\omega,p_1,\dots,p_d)J(\omega)\,d\omega = 2\frac{\Gamma(p_1/2)\dots\Gamma(p_d/2)}{\Gamma(p_1/2+\dots+p_d/2)},$$

where

$$\begin{aligned}\Phi(\omega,p_1,\dots,p_d) &= |\cos\omega_1|^{p_1-1}|\sin\omega_1\cos\omega_2|^{p_2-1}\times\dots\\ &\quad\times|\sin\omega_1\sin\omega_2\dots\sin\omega_{d-2}\cos\omega_{d-1}|^{p_{d-1}-1}\\ &\quad\times|\sin\omega_1\sin\omega_2\dots\sin\omega_{d-2}\sin\omega_{d-1}|^{p_d-1}.\end{aligned}$$

Thus for $d_1(t) = (it)^\alpha$ and $d_2(t) = |t|^n$ we have

$$I = I(2\alpha_1+1,\dots,2\alpha_d+1) = 2\frac{\Gamma(\alpha_1+1/2)\dots\Gamma(\alpha_d+1/2)}{\Gamma(|\alpha|+d/2)},$$

where $|\alpha| = \alpha_1+\cdots\alpha_d$.

Corollary 7. *Let $n > |\alpha| + d/2$. Then*

$$E = C_{d,\alpha}(n)\delta^{\frac{2n-2|\alpha|-d}{2n}},$$

where

$$C_{d,\alpha}(n) = \frac{K_d(|\alpha|,n)}{(2\pi)^{(d-1)/2}}\left(\frac{\Gamma(\alpha_1+1/2)\dots\Gamma(\alpha_d+1/2)}{\Gamma(|\alpha|+d/2)}\right)^{1/2},$$

and the method

$$\widehat{m}(y) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} (it)^\alpha \left(1 + \frac{\delta^2(2|\alpha|+d)}{(2\pi)^d(2n-2|\alpha|-d)}\right)^{-1} y(t)e^{i\langle\tau,t\rangle}\,dt$$

is optimal.

The exact inequality in this case has the form:

$$\|D^\alpha x(\cdot)\|_{L_\infty(\mathbb{R}^d)} \le C_{d,\alpha}(n)\|x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\frac{2n-2|\alpha|-d}{2n}} \|(-\Delta)^{n/2}x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\frac{2|\alpha|+d}{2n}}.$$

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