Optimal Recovery of Operators and Multidimensional Carlson Type Inequalities

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Abstract

The paper is concerned with recovery problems of linear multiplier-type operators from noisy information on weighted classes of functions. Optimal methods of recovery are constructed. The dual extremal problem is closely connected with Carlson type inequalities.

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1. General Setting

Let T be a nonempty set, Σ be the σ-algebra of subsets of T, and μ be a nonnegative σ-additive measure on Σ. We denote by $L_p(T,\Sigma,\mu)$ (or simply $L_p(T,\mu)$) the set of all Σ-measurable functions with values in $\mathbb{R}$ or in $\mathbb{C}$ for which

$$\|x(\cdot)\|_{L_p(T,\mu)} = \left(\int_T |x(t)|^p\,d\mu(t)\right)^{1/p} < \infty, \quad 1 \le p < \infty,$$
$$\|x(\cdot)\|_{L_\infty(T,\mu)} = \operatorname*{ess\,sup}_{t\in T} |x(t)| < \infty, \quad p = \infty.$$

Put

$$\mathcal{W} = \{\, x(\cdot) \in L_p(T,\mu) : \|\varphi(\cdot)x(\cdot)\|_{L_r(T,\mu)} < \infty \,\},$$
$$W = \{\, x(\cdot) \in \mathcal{W} : \|\varphi(\cdot)x(\cdot)\|_{L_r(T,\mu)} \le 1 \,\},$$

where $1 \le p, r \le \infty$, and $\varphi(\cdot)$ is a measurable function on T. Consider the problem of recovery of operator $\Lambda\colon \mathcal{W} \to L_q(T,\mu)$, $1 \le q \le \infty$, defined by equality $\Lambda x(\cdot) = \psi(\cdot)x(\cdot)$, where $\psi(\cdot)$ is a measurable function on T, on the

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class W by the information about functions $x(\cdot) \in W$ given inaccurately. More precisely, we assume that for any function $x(\cdot) \in W$ we know $y(\cdot) \in L_p(T_0, \mu)$, where T_0 is not empty μ-measurable subset of T, such that $\|x(\cdot)-y(\cdot)\|_{L_p(T_0,\mu)} \le \delta$, $\delta \ge 0$. We want to approximate the value $\Lambda x(\cdot)$ knowing $y(\cdot)$.

As recovery methods we consider all possible mappings

$$m\colon L_p(T_0, \mu) \to L_q(T, \mu).$$

The error of a method m is defined as

$$e(p, q, r, m) = \sup_{\substack{x(\cdot)\in W,\ y(\cdot)\in L_p(T_0,\mu)\\ \|x(\cdot)-y(\cdot)\|_{L_p(T_0,\mu)}\le\delta}} \|\Lambda x(\cdot) - m(y)(\cdot)\|_{L_q(T,\mu)}.$$

The quantity

$$E(p, q, r) = \inf_{m\colon L_p(T_0,\mu)\to L_q(T,\mu)} e(p, q, r, m) \tag{1}$$

is known as the optimal recovery error, and a method on which this infimum is attained is called optimal.

Various settings of optimal recovery theory and examples of such problems may be found in [11], [12], [18], [17], [15], [13]. Much of them are devoted to optimal recovery of linear functionals. There are not so many results about optimal recovery of linear operators when non-Euclidean metrics is used ([12, Theorem 12 on p. 45], [10], [14]). In [14] we considered problem (1) when any two of p, q, and r coincide. Here we analyze the case when all metrics can be different and $1 \le q < p, r < \infty$. We construct optimal method of recovery, find its error, and apply this result to obtain exact constants in Carlson type inequalities. The case $p = \infty$ and/or $r = \infty$ requires a slightly different approach. Some particular results of such kind may be found in [7] ($T = \mathbb{Z}$) and [8] ($T = \mathbb{R}$).

2. Main results

Let $\chi_0(\cdot)$ be the characteristic function of the set T_0:

$$\chi_0(t) = \begin{cases} 1, & t \in T_0, \\ 0, & t \notin T_0. \end{cases}$$

Theorem 1. *Let $1 \le q < p, r < \infty$, $\lambda_1, \lambda_2 \ge 0$, $\lambda_1 + \lambda_2 > 0$, $\varphi(t) \ne 0$ for almost all $t \in T \setminus T_0$, $\widehat{x}(t) = \widehat{x}(t, \lambda_1, \lambda_2) \ge 0$ be a solution of equation*

$$-q|\psi(t)|^q + p\lambda_1 x^{p-q}(t)\chi_0(t) + r\lambda_2|\varphi(t)|^r x^{r-q}(t) = 0, \tag{2}$$

and λ_1, λ_2 such that

$$\begin{gathered} \int_{T_0} \widehat{x}^p(t)\, d\mu(t) \le \delta^p, \quad \int_T |\varphi(t)|^r \widehat{x}^r(t)\, d\mu(t) \le 1, \\ \lambda_1\left(\int_{T_0} \widehat{x}^p(t)\, d\mu(t) - \delta^p\right) = 0, \quad \lambda_2\left(\int_T |\varphi(t)|^r \widehat{x}^r(t)\, d\mu(t) - 1\right) = 0, \end{gathered} \tag{3}$$

and $\lambda_2 > 0$, *if* $T \setminus T_0 \neq \emptyset$. *Then*

$$E(p,q,r) = \left(\frac{p\lambda_1\delta^p + r\lambda_2}{q}\right)^{1/q},$$

and the method

$$\widehat{m}(y)(t) = \begin{cases} q^{-1}p\lambda_1\widehat{x}^{p-q}(t)|\psi(t)|^{-q}\psi(t)y(t), & t \in T_0,\ \psi(t) \neq 0, \\ 0, & \textit{otherwise}, \end{cases} \tag{4}$$

is optimal recovery method.

To prove this theorem we need some preliminary results.

Lemma 1.

$$E(p,q,r) \geq \sup_{\substack{x(\cdot)\in W \\ \|x(\cdot)\|_{L_p(T_0,\mu)}\leq\delta}} \|\Lambda x(\cdot)\|_{L_q(T,\mu)}. \tag{5}$$

The lower bound of type (5) is the well-known result which is usually applied to obtain the error of optimal recovery. In more or less general forms it was proved in many papers (see, for example, [14]).

The extremal problem which arises on the right-hand side of (5), known as the dual problem, may be written as

$$\|\psi(\cdot)x(\cdot)\|_{L_q(T,\mu)} \to \max, \quad \|x(\cdot)\|_{L_p(T_0,\mu)} \leq \delta,$$
$$\|\varphi(\cdot)x(\cdot)\|_{L_r(T,\mu)} \leq 1. \tag{6}$$

For $T_0 = T \subset \mathbb{R}^n$ and $q = 1$ problem (6) was examined in [2] in connection with Stechkin's problem.

We give a straightforward result (resembling the sufficient conditions in the Kuhn-Tucker theorem), which we will require in solving dual problems similar to (6).

Let $f_j\colon A \to \mathbb{R}$, $j = 0, 1, \ldots, n$, be functions defined on some set A. Consider the extremal problem

$$f_0(x) \to \max, \quad f_j(x) \leq 0, \quad j = 1, \ldots, n, \quad x \in A, \tag{7}$$

and write down its Lagrange function

$$\mathcal{L}(x,\lambda) = -f_0(x) + \sum_{j=1}^{n} \lambda_j f_j(x), \quad \lambda = (\lambda_1, \ldots, \lambda_n).$$

Lemma 2 ([14])**.** *Assume that there exist* $\widehat{\lambda}_j \geq 0$, $j = 1, \ldots, n$, *and an element* $\widehat{x} \in A$, *admissible for problem* (7), *such that*

$$(a) \quad \min_{x\in A} \mathcal{L}(x,\widehat{\lambda}) = \mathcal{L}(\widehat{x},\widehat{\lambda}), \quad \widehat{\lambda} = (\widehat{\lambda}_1, \ldots, \widehat{\lambda}_n),$$

$$(b) \quad \sum_{j=1}^{n} \widehat{\lambda}_j f_j(\widehat{x}) = 0.$$

Then $\widehat{x}$ *is an extremal element for problem* (7).

Put

$$F(u,v,\alpha) = -((1-\alpha)u+\alpha v)^q + av^p + bu^r, \quad u,v \ge 0, \quad \alpha \in [0,1],$$

where $a,b \ge 0$, and $1 \le p,q,r < \infty$.

Lemma 3. *For all $a,b \ge 0$, $a+b>0$, and all $1 \le q < p,r < \infty$, there exists the unique solution $\widehat{u} > 0$ of the equation*

$$-q + pau^{p-q} + rbu^{r-q} = 0. \tag{8}$$

Moreover, for all $u,v \ge 0$ and $\alpha = q^{-1}pa\widehat{u}^{p-q} = 1 - q^{-1}rb\widehat{u}^{r-q}$

$$F(\widehat{u},\widehat{u},\alpha) \le F(u,v,\alpha). \tag{9}$$

In particular, for all $u \ge 0$

$$-\widehat{u}^q + a\widehat{u}^p + b\widehat{u}^r \le -u^q + au^p + bu^r.$$

Proof. The existence of the unique solution of (8) follows from the fact that the continuous function $f(u) = pau^{p-q} + rbu^{r-q}$ increases monotonically from 0 to $+\infty$.

Let us prove (9). The cases $a=0$ or $b=0$ are easily obtained by finding the minimum of $F(u,v,0) = -u^q + bu^r$ if $a=0$ or $F(u,v,1) = -v^q + av^p$ if $b=0$. Assume that $a,b>0$. Then $\alpha \in (0,1)$. Let

$$C > \max\{a^{-\frac{1}{p-q}}, b^{-\frac{1}{r-q}}\}.$$

Then for $u \ge C$ and $v \le u$ we have

$$F(u,v,\alpha) \ge -u^q + bu^r = u^q(-1+bu^{r-q}) > 0. \tag{10}$$

If $v \ge C$ and $v \ge u$, then

$$F(u,v,\alpha) \ge -v^q + av^p = v^q(-1+av^{p-q}) > 0. \tag{11}$$

Since $F(0,0,\alpha)=0$ we obtain that

$$\inf_{(u,v)\in\mathbb{R}^2_+} F(u,v,\alpha) = \inf_{\substack{0\le u\le C\\ 0\le v\le C}} F(u,v,\alpha).$$

It follows from the Weierstrass extreme value theorem that there exist $0 \le u_0 \le C$ and $0 \le v_0 \le C$ such that

$$\inf_{(u,v)\in\mathbb{R}^2_+} F(u,v,\alpha) = F(u_0,v_0,\alpha).$$

In view of (10) and (11) $u_0 < C$ and $v_0 < C$. We have

$$\begin{aligned} F_u(u,v,\alpha) &= -q((1-\alpha)u+\alpha v)^{q-1}(1-\alpha) + rbu^{r-1} \\ &= rb(-((1-\alpha)u+\alpha v)^{q-1}\widehat{u}^{r-q} + u^{r-1}). \end{aligned}$$

Thus, for any $v_0 \geq 0$ and sufficiently small $u > 0$ $F_u(u, v_0, \alpha) < 0$. Consequently,

$$F(u, v_0, \alpha) < F(0, v_0, \alpha)$$

for sufficiently small u. It means that $0 < u_0 < C$. The similar arguments show that $0 < v_0 < C$. Hence

$$F_u(u_0, v_0, \alpha) = F_v(u_0, v_0, \alpha) = 0.$$

Since

$$F_v(u, v, \alpha) = -q((1-\alpha)u + \alpha v)^{q-1}\alpha + pav^{p-1}$$
$$= pa(-((1-\alpha)u + \alpha v)^{q-1}\widehat{u}^{p-q} + v^{p-1})$$

we have

$$-((1-\alpha)u_0 + \alpha v_0)^{q-1}\widehat{u}^{r-q} + u_0^{r-1} = 0, \tag{12}$$
$$-((1-\alpha)u_0 + \alpha v_0)^{q-1}\widehat{u}^{p-q} + v_0^{p-1} = 0. \tag{13}$$

Consequently,

$$\frac{u_0^{r-1}}{v_0^{p-1}} = \widehat{u}^{r-p}.$$

Suppose that $p \leq r$. Substituting

$$u_0 = \widehat{u}^{\frac{r-p}{r-1}} v_0^{\frac{p-1}{r-1}} \tag{14}$$

into (13), we obtain the equality

$$(\alpha v_0 + (1-\alpha)\widehat{u}^{\frac{r-p}{r-1}} v_0^{\frac{p-1}{r-1}})^{q-1}\widehat{u}^{p-q} = v_0^{p-1}.$$

This equality may be rewritten in the form

$$(\alpha + (1-\alpha)t^{\frac{p-r}{r-1}})^{q-1} = t^{p-q}, \tag{15}$$

where $t = v_0\widehat{u}^{-1}$. It is easily seen that (15) has the unique solution $t = 1$. Consequently, $v_0 = \widehat{u}$ and it follows by (14) that $u_0 = \widehat{u}$.

If $p > r$, then we substitute

$$v_0 = \widehat{u}^{\frac{p-r}{p-1}} u_0^{\frac{r-1}{p-1}}$$

into (12). Similar to the previous case we obtain the equality which may be written in the form

$$(\alpha s^{\frac{r-p}{p-1}} + 1 - \alpha)^{q-1} = s^{r-q}, \tag{16}$$

where $s = u_0\widehat{u}^{-1}$. The unique solution of (16) is $s = 1$. Thus, for the case when $p > r$ we have the same solution of (12), (13) $u_0 = v_0 = \widehat{u}$. Hence, for all $u, v \geq 0$

$$F(u, v, \alpha) \geq \inf_{(u,v)\in\mathbb{R}_+^2} F(u, v, \alpha) = F(\widehat{u}, \widehat{u}, \alpha).$$

□

Proof of Theorem 1.

1. Lower estimate. The extremal problem (6) (for convenience, we raise the quantity to be maximized to the q-th power) is as follows:

$$\int_T |\psi(t)x(t)|^q\,d\mu(t) \to \max, \quad \int_{T_0} |x(t)|^p\,d\mu(t) \le \delta^p,$$
$$\int_T |\varphi(t)x(t)|^r\,d\mu(t) \le 1. \qquad (17)$$

The Lagrange function for this problem reads as

$$\mathcal{L}(x(\cdot),\lambda_1,\lambda_2) = \int_T L\big(t,x(t),\lambda_1,\lambda_2\big)\,d\mu(t),$$

where

$$L(t,x,\lambda_1,\lambda_2) = -|\psi(t)x|^q + \lambda_1|x|^p\chi_0(t) + \lambda_2|\varphi(t)x|^r.$$

If $t \in T$ such that $\psi(t) = 0$, then evidently $\widehat{x}(t) = 0$ and for those t for all $x(\cdot) \in \mathcal{W}$

$$L(t,0,\lambda_1,\lambda_2) \le L(t,x(t),\lambda_1,\lambda_2).$$

Using this fact and Lemma 3, we obtain that there is the unique solution $\widehat{x}(\cdot)$ of (2) and, moreover, for almost all $t \in T$ and all $x(\cdot) \in \mathcal{W}$

$$L(t,\widehat{x}(t),\lambda_1,\lambda_2) \le L(t,x(t),\lambda_1,\lambda_2).$$

Consequently,

$$\mathcal{L}(\widehat{x}(\cdot),\lambda_1,\lambda_2) \le \mathcal{L}(x(\cdot),\lambda_1,\lambda_2).$$

Taking into account (3) we obtain by Lemma 2 that $\widehat{x}(\cdot)$ is the extremal function in (17). It follows by (5) that

$$E(p,q,r) \ge \left(\int_T |\psi(t)|^q \widehat{x}^q(t)\,d\mu(t)\right)^{1/q}.$$

From (2) we have

$$|\psi(t)|^q \widehat{x}^q(t) = q^{-1}p\lambda_1\widehat{x}^p(t)\chi_{T_0}(t) + q^{-1}r\lambda_2|\varphi(t)|^r\widehat{x}^r(t).$$

Integrating this equality over the set T, we obtain

$$\int_T |\psi(t)|^q \widehat{x}^q(t)\,d\mu(t) = \frac{p\lambda_1\delta^p + r\lambda_2}{q}. \qquad (18)$$

Thus,

$$E(p,q,r) \ge \left(\frac{p\lambda_1\delta^p + r\lambda_2}{q}\right)^{1/q}.$$

2. Upper estimate. To estimate the error of method (4) we need to find the value of the extremal problem:

$$\int_{T_0} |\psi(t)x(t) - \psi(t)\alpha(t)y(t)|^q \, d\mu(t) + \int_{T\setminus T_0} |\psi(t)x(t)|^q \, d\mu(t) \to \max,$$
$$\int_{T_0} |x(t) - y(t)|^p \, d\mu(t) \le \delta^p, \quad \int_T |\varphi(t)x(t)|^r \, d\mu(t) \le 1, \quad (19)$$

where

$$\alpha(t) = \begin{cases} q^{-1}p\lambda_1 \widehat{x}^{p-q}(t)|\psi(t)|^{-q}, & t \in T_0, \ \psi(t) \ne 0, \\ 0, & \text{otherwise}. \end{cases} \quad (20)$$

Taking

$$z(t) = \begin{cases} x(t) - y(t), & t \in T_0, \\ 0, & t \in T \setminus T_0, \end{cases}$$

we rewrite (19) as follows:

$$\int_T |\psi(t)|^q |(1-\alpha(t))x(t) + \alpha(t)z(t)|^q \, d\mu(t) \to \max,$$
$$\int_{T_0} |z(t)|^p \, d\mu(t) \le \delta^p, \quad \int_T |\varphi(t)x(t)|^r \, d\mu(t) \le 1.$$

The value of this problem does not exceed the value of the problem

$$\int_T |\psi(t)|^q ((1-\alpha(t))u(t) + \alpha(t)v(t))^q \, d\mu(t) \to \max,$$
$$\int_{T_0} v^p(t) \, d\mu(t) \le \delta^p, \quad \int_T |\varphi(t)|^r u^r(t) \, d\mu(t) \le 1,$$
$$u(t) \ge 0, \ v(t) \ge 0 \quad \text{for almost all} \ \ t \in T. \quad (21)$$

The Lagrange function for this problem is

$$\mathcal{L}_1(u(\cdot), v(\cdot), \mu_1, \mu_2) = \int_T L_1(t, u(t), v(t), \mu_1, \mu_2) \, d\mu(t),$$

where

$$L_1(t, u, v, \mu_1, \mu_2) = -|\psi(t)|^q ((1-\alpha(t))u + \alpha(t)v)^q$$
$$+ \mu_1 v^p \chi_0(t) + \mu_2 |\varphi(t)|^r u^r.$$

By Lemma 3 we have

$$L_1(t, \widehat{x}(t), \widehat{x}(t), \lambda_1, \lambda_2) \le L_1(t, u(t), v(t), \lambda_1, \lambda_2).$$

Thus,

$$\mathcal{L}_1(\widehat{x}(\cdot), \widehat{x}(\cdot), \lambda_1, \lambda_2) \le \mathcal{L}_1(u(\cdot), v(\cdot), \lambda_1, \lambda_2).$$

It follows by Lemma 2 that functions $u(t) = v(t) = \widehat{x}(t)$ are extremal in (21). Consequently,

$$e(p,q,r,\widehat{m}) \le \left(\int_T |\psi(t)|^q \widehat{x}^q(t)\, d\mu(t)\right)^{1/q} = \left(\frac{p\lambda_1\delta^p + r\lambda_2}{q}\right)^{1/q} \le E(p,q,r).$$

It means that the method (4) is optimal and the optimal recovery error is as stated. $\square$

Note that if conditions of Theorem 1 hold we proved the equality

$$E(p,q,r) = \sup_{\substack{\|x(\cdot)\|_{L_p(T_0,\mu)} \le \delta \\ \|\varphi(\cdot)x(\cdot)\|_{L_r(T,\mu)} \le 1}} \|\psi(\cdot)x(\cdot)\|_{L_q(T,\mu)}. \tag{22}$$

Corollary 1. *Let $1 \le q < p, r < \infty$, $\varphi(t) \ne 0$ for almost all $t \in T$, and*

$$0 < \int_T \left|\frac{\psi(t)}{\varphi(t)}\right|^{\frac{qr}{r-q}} d\mu(t) < \infty, \quad \int_{T_0} \left(\frac{|\psi(t)|^q}{|\varphi(t)|^r}\right)^{\frac{p}{r-q}} d\mu(t) < \infty.$$

Then for all

$$\delta \ge \frac{\left(\displaystyle\int_{T_0} \left(\frac{|\psi(t)|^q}{|\varphi(t)|^r}\right)^{\frac{p}{r-q}} d\mu(t)\right)^{1/p}}{\left(\displaystyle\int_T \left|\frac{\psi(t)}{\varphi(t)}\right|^{\frac{qr}{r-q}} d\mu(t)\right)^{1/r}}$$

$$E(p,q,r) = \left(\int_T \left|\frac{\psi(t)}{\varphi(t)}\right|^{\frac{qr}{r-q}} d\mu(t)\right)^{\frac{r-q}{qr}},$$

and the method $\widehat{m}(y)(t) = 0$ is optimal recovery method.

Proof. It suffices to check that $\lambda_1 = 0$ and

$$\lambda_2 = \frac{q}{r}\left(\int_T \left|\frac{\psi(t)}{\varphi(t)}\right|^{\frac{qr}{r-q}} d\mu(t)\right)^{\frac{r-q}{r}}$$

satisfy the conditions of Theorem 1. $\square$

Corollary 2. *Let $1 \le q < p, r < \infty$, $T_0 = T$, and*

$$0 < \int_T |\varphi(t)|^r |\psi(t)|^{\frac{qr}{p-q}} d\mu(t) < \infty, \quad \int_T |\psi(t)|^{\frac{qp}{p-q}} d\mu(t) < \infty.$$

Then for all

$$\delta \le \frac{\left(\displaystyle\int_T |\psi(t)|^{\frac{qp}{p-q}} d\mu(t)\right)^{1/p}}{\left(\displaystyle\int_T |\varphi(t)|^r |\psi(t)|^{\frac{qr}{p-q}} d\mu(t)\right)^{1/r}}$$

$$E(p,q,r) = \delta\left(\int_T |\psi(t)|^{\frac{qp}{p-q}} d\mu(t)\right)^{\frac{p-q}{qp}},$$

and the method $\widehat{m}(y)(t) = \psi(t)y(t)$ *is optimal recovery method.*

Proof. It suffices to check that

$$\lambda_1 = \frac{q}{p\delta^{p-q}}\left(\int_T |\psi(t)|^{\frac{qp}{p-q}}\,d\mu(t)\right)^{\frac{p-q}{p}}$$

and $\lambda_2 = 0$ satisfy the conditions of Theorem 1. □

Note that assumption (3) need not be satisfied in all cases. For example, in the trivial case $\delta = 0$, $T_0 = T$, and $\psi(t) = 1$ there are no such λ_1 and λ_2 which satisfy (3).

Let us consider the problem of optimal recovery of the linear functional

$$Lx = \int_T \psi(t)x(t)\,d\mu(t)$$

on the class W, knowing $y(\cdot) \in L_p(T_0,\mu)$, $T_0 \subset T$, such that $\|x(\cdot) - y(\cdot)\|_{L_p(T_0,\mu)} \le \delta$, $\delta \ge 0$. In this case as recovery methods we consider all possible mappings $m\colon L_p(T_0,\mu) \to \mathbb{C}$ or $\mathbb{R}$. The error of a method m is defined as

$$e_1(p,r,m) = \sup_{\substack{x(\cdot)\in W,\ y(\cdot)\in L_p(T_0,\mu)\\ \|x(\cdot)-y(\cdot)\|_{L_p(T_0,\mu)}\le\delta}} |Lx - m(y)|.$$

The quantity

$$E_1(p,r) = \inf_{m\colon L_p(T_0,\mu)\to\mathbb{C}(\mathbb{R})} e_1(q,r,m) \tag{23}$$

is optimal recovery error, and a method on which this infimum is attained is called optimal.

Theorem 1′. *Let* $1 < p, r < \infty$, $\lambda_1, \lambda_2 \ge 0$, $\lambda_1 + \lambda_2 > 0$, $\varphi(t) \ne 0$ *for almost all* $t \in T \setminus T_0$, $\widehat{x}(t) = \widehat{x}(t,\lambda_1,\lambda_2) \ge 0$ *be a solution of equation*

$$-|\psi(t)| + p\lambda_1 x^{p-1}(t)\chi_0(t) + r\lambda_2|\varphi(t)|^r x^{r-1}(t) = 0,$$

and λ_1, λ_2 *such that conditions* (3) *are fulfilled, and* $\lambda_2 > 0$, *if* $T\setminus T_0 \ne \emptyset$. *Then*

$$E_1(p,r) = p\lambda_1\delta^p + r\lambda_2,$$

and the method

$$\widehat{m}(y) = p\lambda_1 \int_{T_0} \widehat{x}^{p-1}(t)\varepsilon(t)y(t)\,d\mu(t), \tag{24}$$

where

$$\varepsilon(t) = \begin{cases} \dfrac{\psi(t)}{|\psi(t)|}, & \psi(t) \ne 0,\\ 1, & \psi(t) = 0, \end{cases}$$

is optimal recovery method.

Proof. For the functional case it is known (see, for example, [6]) that

$$E_1(p,r) = \sup_{\substack{x(\cdot)\in W \\ \|x(\cdot)\|_{L_p(T_0,\mu)}\le\delta}} \left|\int_T \psi(t)x(t)\,d\mu(t)\right|.$$

Put $\widetilde{x}(\cdot) = \overline{\varepsilon(\cdot)}\widehat{x}(\cdot)$. It follows by (3) that $\widetilde{x}(\cdot) \in W$ and $\|\widetilde{x}(\cdot)\|_{L_p(T_0,\mu)} \le \delta$. Taking into account (18), we obtain

$$E_1(p,r) \ge \left|\int_T \psi(t)\widetilde{x}(t)\,d\mu(t)\right| = \int_T |\psi(t)|\widehat{x}(t)\,d\mu(t) = p\lambda_1\delta^p + r\lambda_2.$$

Now we estimate the error of method (24). We have

$$\begin{aligned} e_1(p,r,\widehat{m}) &= \sup_{\substack{x(\cdot)\in W,\ y(\cdot)\in L_p(T_0,\mu) \\ \|x(\cdot)-y(\cdot)\|_{L_p(T_0,\mu)}\le\delta}} \left|\int_T \psi(t)x(t)\,d\mu(t) - \widehat{m}(y)\right| \\ &\le \sup_{\substack{x(\cdot)\in W,\ z(\cdot)\in L_p(T_0,\mu) \\ \|z(\cdot)\|_{L_p(T_0,\mu)}\le\delta}} \int_T |\psi(t)||(1-\alpha(t))x(t) + \alpha(t)z(t)|\,d\mu(t), \end{aligned}$$

where $\alpha(\cdot)$ is defined by (20) for $q=1$. It follows from the proof of Theorem 1 that

$$E_1(p,r) \le e_1(p,r,\widehat{m}) \le \int_T |\psi(t)|\widehat{x}(t)\,d\mu(t) = p\lambda_1\delta^p + r\lambda_2.$$

□

One can easily obtain analogs of Corollaries 1 and 2 for problem (23).

3. The case of homogenous weight functions

Let T be a cone in a linear space, $T_0 = T$, $|\psi(\cdot)|$ and $|\varphi(\cdot)|$ be homogenous functions of degrees η, ν, respectively, $\varphi(t) \ne 0$ and $\psi(t) \ne 0$ for almost all $t \in T$, and $\mu(\cdot)$ be a homogenous measure of degree d. We assume, again, that $1 \le p < q, r < \infty$. For $k \in [0,1)$ the function $k^{\frac{1}{p-q}}(1-k)^{-\frac{1}{r-q}}$ increases monotonically from 0 to $+\infty$. Consequently, for all $z \in T$ such that $\varphi(z) \ne 0$ and $\psi(z) \ne 0$ (if $p < r$), there exists $k(z)$ for which

$$\frac{k^{\frac{1}{p-q}}(z)}{(1-k(z))^{\frac{1}{r-q}}} = \frac{|\psi(z)|^{\frac{q(p-r)}{(p-q)(r-q)}}}{|\varphi(z)|^{\frac{r}{r-q}}}. \tag{25}$$

Thus, the function $k(z)$ is well defined for almost all $z \in T$.

Theorem 2. *Let $1 \le q < p, r < \infty$, $\varphi(t), \psi(t) \neq 0$ for almost all $t \in T$, and $\nu + d(1/r - 1/p) \neq 0$. Assume that*

$$I_1 = \int_T |\psi(z)|^{\frac{qp}{p-q}} k^{\frac{p}{p-q}}(z)\, d\mu(z) < \infty,$$
$$I_2 = \int_T |\psi(z)|^{\frac{qr}{p-q}} |\varphi(z)|^r k^{\frac{r}{p-q}}(z)\, d\mu(z) < \infty.$$

Then

$$E(p,q,r) = \delta^\gamma I_1^{-\gamma/p} I_2^{-(1-\gamma)/r} (I_1 + I_2)^{1/q},$$

where

$$\gamma = \frac{\nu - \eta - d(1/q - 1/r)}{\nu + d(1/r - 1/p)}, \tag{26}$$

and the method

$$\widehat{m}(y)(t) = k(\xi t)\psi(t)y(t),$$

where

$$\xi = \left(\delta I_1^{-1/p} I_2^{1/r}\right)^{\frac{1}{\nu + d(1/r - 1/p)}}, \tag{27}$$

is optimal recovery method.

Proof. Put

$$\widehat{x}(t) = \left(\frac{q|\psi(t)|^q}{p\lambda_1}\right)^{\frac{1}{p-q}} k^{\frac{1}{p-q}}(\xi t),$$

where $\lambda_1 > 0$ will be specified later. We show that $\widehat{x}(\cdot)$ satisfies (2), where

$$\lambda_2 = r^{-1} q^{\frac{p-r}{p-q}} (p\lambda_1)^{\frac{r-q}{p-q}} \xi^{\nu r - \eta \frac{q(p-r)}{p-q}}. \tag{28}$$

We have

$$p\lambda_1 \widehat{x}^{p-q}(t) = q|\psi(t)|^q k(\xi t),$$

and further,

$$r\lambda_2 |\varphi(t)|^r \widehat{x}^{r-q}(t) = r\lambda_2 |\varphi(t)|^r \left(\frac{q|\psi(t)|^q}{p\lambda_1}\right)^{\frac{r-q}{p-q}} k^{\frac{r-q}{p-q}}(\xi t).$$

Since $|\varphi(\cdot)|$ and $|\psi(\cdot)|$ are homogenous it follows by (25) that

$$k^{\frac{r-q}{p-q}}(\xi t) = \frac{|\psi(\xi t)|^{\frac{q(p-r)}{p-q}}}{|\varphi(\xi t)|^r}(1 - k(\xi t)) = \xi^{\eta\frac{q(p-r)}{p-q} - \nu r} \frac{|\psi(t)|^{\frac{q(p-r)}{p-q}}}{|\varphi(t)|^r}(1 - k(\xi t)).$$

Thus,

$$r\lambda_2 |\varphi(t)|^r \widehat{x}^{r-q}(t) = r\lambda_2 \left(\frac{q}{p\lambda_1}\right)^{\frac{r-q}{p-q}} \xi^{\eta\frac{q(p-r)}{p-q} - \nu r} |\psi(t)|^q (1 - k(\xi t))$$
$$= q|\psi(t)|^q (1 - k(\xi t)) = q|\psi(t)|^q - p\lambda_1 \widehat{x}^{p-q}(t).$$

Now we show that for

$$\lambda_1 = \frac{q}{p} I_1^{\frac{p-q}{p}} \xi^{-\eta q - d\frac{p-q}{p}} \delta^{q-p} \tag{29}$$

the equalities

$$\int_T \widehat{x}^p(t)\, d\mu(t) = \delta^p, \quad \int_T |\varphi(t)|^r \widehat{x}^r(t)\, d\mu(t) = 1$$

hold. In view of the definition of $\widehat{x}(\cdot)$ we need to check that

$$\int_T \left(\frac{q|\psi(t)|^q}{p\lambda_1}\right)^{\frac{p}{p-q}} k^{\frac{p}{p-q}}(\xi t)\, d\mu(t) = \delta^p,$$
$$\int_T |\varphi(t)|^r \left(\frac{q|\psi(t)|^q}{p\lambda_1}\right)^{\frac{r}{p-q}} k^{\frac{r}{p-q}}(\xi t)\, d\mu(t) = 1.$$

Changing $z = \xi t$ and taking into account that functions $|\psi(\cdot)|$, $|\varphi(\cdot)|$ with the measure $\mu(\cdot)$ are homogenous, we obtain

$$\left(\frac{q}{p\lambda_1}\right)^{\frac{p}{p-q}} I_1 = \delta^p \xi^{\frac{\eta q p}{p-q}+d},$$
$$\left(\frac{q}{p\lambda_1}\right)^{\frac{r}{p-q}} I_2 = \xi^{\frac{\eta q r}{p-q}+\nu r+d}.$$

The validity of these equalities immediately follows from the definitions of λ_1 and ξ.

It follows by Theorem 1, (29), (28), and (27) that

$$E^q(p,q,r) = \frac{p\lambda_1\delta^p + r\lambda_2}{q} = I_1^{\frac{p-q}{p}} \xi^{-\eta q - d\frac{p-q}{p}} \delta^q + \left(\frac{p\lambda_1}{q}\right)^{\frac{r-q}{p-q}} \xi^{\nu r - \eta\frac{q(p-r)}{p-q}}$$
$$= \delta^{q\gamma} I_1^{-q\gamma/p} I_2^{-q(1-\gamma)/r}(I_1+I_2).$$

Moreover, the same theorem states that the method

$$\widehat{m}(y)(t) = q^{-1}p\lambda_1\widehat{x}^{p-q}(t)|\psi(t)|^{-q}\psi(t)y(t) = k(\xi t)\psi(t)y(t)$$

is optimal. □

It follows by Theorem 2 and (22) that for all $x(\cdot) \in \mathcal{W}$ such that $\|\varphi(\cdot)x(\cdot)\|_{L_r(T,\mu)} \le 1$ the exact inequality

$$\|\psi(\cdot)x(\cdot)\|_{L_q(T,\mu)} \le C\|x(\cdot)\|_{L_p(T,\mu)}^\gamma \tag{30}$$

holds, where

$$C = I_1^{-\gamma/p} I_2^{-(1-\gamma)/r}(I_1+I_2)^{1/q}.$$

(Here and later the exactness means that C cannot be replaced by any other constant smaller than C).

From (30) the following exact inequality can be easily obtained

$$\|\psi(\cdot)x(\cdot)\|_{L_q(T,\mu)} \le C\|x(\cdot)\|^{\gamma}_{L_p(T,\mu)}\|\varphi(\cdot)x(\cdot)\|^{1-\gamma}_{L_r(T,\mu)}, \tag{31}$$

which holds for all $x(\cdot) \in \mathcal{W}$, $x(\cdot) \ne 0$.

Let $|w(\cdot)|$, $|w_0(\cdot)|$, and $|w_1(\cdot)|$ be homogenous functions of degrees θ, θ_0, and θ_1, respectively. We assume that $w(t), w_0(t), w_1(t) \ne 0$ for almost all $t \in T$ and $1 \le q < p, r < \infty$. Then for almost all $z \in T$ such that $w(z), w_0(z), w_1(z) \ne 0$ there exists $\widetilde{k}(z)$ satisfying

$$\frac{\widetilde{k}^{\frac{1}{p-q}}(z)}{(1-\widetilde{k}(z))^{\frac{1}{r-q}}} = \left|\frac{w(z)}{w_1(z)}\right|^{\frac{r}{r-q}} \left|\frac{w_0(z)}{w(z)}\right|^{\frac{p}{p-q}}.$$

Put

$$\widetilde{\theta} = \theta + d/q, \quad \widetilde{\theta}_0 = \theta_0 + d/p, \quad \widetilde{\theta}_1 = \theta_1 + d/r. \tag{32}$$

Corollary 3. *Let $1 \le q < p, r < \infty$, $w(t), w_0(t), w_1(t) \ne 0$ for almost all $t \in T$, and $\widetilde{\theta}_0 \ne \widetilde{\theta}_1$. Assume that*

$$\widetilde{I}_1 = \int_T \left|\frac{w(z)}{w_0(z)}\right|^{\frac{qp}{p-q}} \widetilde{k}^{\frac{p}{p-q}}(z)\,d\mu(z) < \infty,$$

$$\widetilde{I}_2 = \int_T \frac{|w(z)|^{\frac{qr}{p-q}}}{|w_0(z)|^{\frac{pr}{p-q}}}|w_1(z)|^r \widetilde{k}^{\frac{r}{p-q}}(z)\,d\mu(z) < \infty.$$

Then for all $x(\cdot) \ne 0$ such that $w_0(\cdot)x(\cdot) \in L_p(T,\mu)$ and $w_1(\cdot)x(\cdot) \in L_r(T,\mu)$ the exact inequality

$$\|w(\cdot)x(\cdot)\|_{L_q(T,\mu)} \le \widetilde{C}\|w_0(\cdot)x(\cdot)\|^{\widetilde{\gamma}}_{L_p(T,\mu))}\|w_1(\cdot)x(\cdot)\|^{1-\widetilde{\gamma}}_{L_r(T,\mu)} \tag{33}$$

holds; here

$$\widetilde{C} = \widetilde{I}_1^{-\widetilde{\gamma}/p}\widetilde{I}_2^{-(1-\widetilde{\gamma})/r}(\widetilde{I}_1 + \widetilde{I}_2)^{1/q}, \quad \widetilde{\gamma} = \frac{\widetilde{\theta}_1 - \widetilde{\theta}}{\widetilde{\theta}_1 - \widetilde{\theta}_0}.$$

Proof. Put

$$\psi(x) = \frac{w(x)}{w_0(x)}, \quad \varphi(x) = \frac{w_1(x)}{w_0(x)}.$$

Then $|\psi(\cdot)|$ and $|\varphi(\cdot)|$ are homogeneous functions of degrees $\eta = \theta - \theta_0$ and $\nu = \theta_1 - \theta_0$, respectively. It follows by (31) that for all $x(\cdot) \in \mathcal{W}$, $x(\cdot) \ne 0$, the exact inequality

$$\|\psi(\cdot)x(\cdot)\|_{L_q(T,\mu)} \le \widetilde{C}\|x(\cdot)\|^{\widetilde{\gamma}}_{L_p(T,\mu)}\|\varphi(\cdot)x(\cdot)\|^{1-\widetilde{\gamma}}_{L_r(T,\mu)}$$

holds. Substituting $x(\cdot) = w_0(\cdot)y(\cdot)$, we obtain (33). □

The well-known Carlson inequality [4]

$$\|x(t)\|_{L_1(\mathbb{R}_+)} \le \sqrt{\pi}\|x(t)\|^{1/2}_{L_2(\mathbb{R}_+)}\|tx(t)\|^{1/2}_{L_2(\mathbb{R}_+)} \tag{34}$$

was generalized in many directions (see [5], [1], [3]). Inequality (33) is also a generalization of the Carlson inequality.

Let $1 \le p < q, r < \infty$, T be a cone in $\mathbb{R}^d$, $d\mu(t) = dt$, $|\psi(\cdot)|$ and $|\varphi(\cdot)|$ be homogenous functions of degrees η, ν, respectively, $\varphi(t) \ne 0$ and $\psi(t) \ne 0$ for almost all $t \in T$. Thus $\mu(\cdot)$ is a homogeneous measure of degree d. Consider the polar transformation

$$\begin{aligned}
x_1 &= \rho\cos\omega_1,\\
x_2 &= \rho\sin\omega_1\cos\omega_2,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
x_{d-1} &= \rho\sin\omega_1\sin\omega_2\dots\sin\omega_{d-2}\cos\omega_{d-1},\\
x_d &= \rho\sin\omega_1\sin\omega_2\dots\sin\omega_{d-2}\sin\omega_{d-1}.
\end{aligned}$$

Set $\omega = (\omega_1, \dots, \omega_{d-1})$,

$$\begin{aligned}
\widetilde{\psi}(\omega) &= \rho^{-\eta}|\psi(\rho\cos\omega_1, \dots, \rho\sin\omega_1\sin\omega_2\dots\sin\omega_{d-2}\sin\omega_{d-1})|,\\
\widetilde{\varphi}(\omega) &= \rho^{-\nu}|\varphi(\rho\cos\omega_1, \dots, \rho\sin\omega_1\sin\omega_2\dots\sin\omega_{d-2}\sin\omega_{d-1})|.
\end{aligned} \tag{35}$$

Denote by Ω the range of ω. Since T is a cone, Ω does not depend on ρ. Put

$$J(\omega) = \sin^{d-2}\omega_1\sin^{d-3}\omega_2\dots\sin\omega_{d-2}.$$

By (25) we obtain the following equality for $k(\cdot)$:

$$\frac{k^{\frac{1}{p-q}}(\rho,\omega)}{(1-k(\rho,\omega))^{\frac{1}{r-q}}} = \rho^{\frac{\eta q(p-r)-\nu r(p-q)}{(p-q)(r-q)}}\frac{\widetilde{\psi}^{\frac{q(p-r)}{(p-q)(r-q)}}(\omega)}{\widetilde{\varphi}^{\frac{r}{r-q}}(\omega)}. \tag{36}$$

Assume that $\gamma \in (0,1)$, where γ is defined by (26). Put

$$\frac{1}{q^*} = \frac{1}{q} - \frac{\gamma}{p} - \frac{1-\gamma}{r}. \tag{37}$$

It is easy to verify that $q^* > q \ge 1$. Moreover,

$$q^* = \frac{pqr(\nu + d(1/r - 1/p))}{\nu r(p-q) - \eta q(p-r)}.$$

Theorem 3. *Let $1 \le q < p, r < \infty$, $\gamma \in (0,1)$, and $\widetilde{\varphi}(\omega), \widetilde{\psi}(\omega) \ne 0$ for almost all $\omega \in \Omega$. Assume that*

$$I = \int_\Omega \frac{\widetilde{\psi}^{q^*}(\omega)}{\widetilde{\varphi}^{q^*(1-\gamma)}(\omega)}J(\omega)\,d\omega < \infty.$$

Then

$$E(p,q,r) = C_1\delta^\gamma,$$

where

$$C_1 = \gamma^{-\frac{\gamma}{p}}(1-\gamma)^{-\frac{1-\gamma}{r}}\left(\frac{B\left(q^*\gamma/p, q^*(1-\gamma)/r\right)I}{|\nu + d(1/r-1/p)|(\gamma r + (1-\gamma)p)}\right)^{1/q^*},$$

where $B(\cdot,\cdot)$ is the beta-function. Moreover, the method

$$\widehat{m}(y)(t) = k\left(\xi_1^{\frac{1}{\nu+d(1/r-1/p)}}t\right)\psi(t)y(t),$$

where

$$\xi_1 = \delta\left(\gamma^{q-r}(1-\gamma)^{p-q}C_1^{p-r}\right)^{\frac{q^*}{pqr}},$$

is optimal recovery method.

Proof. Using Theorem 2, we obtain

$$I_1 = \int_T |\psi(z)|^{\frac{qp}{p-q}} k^{\frac{p}{p-q}}(z)\,dz$$
$$= \int_\Omega \widetilde{\psi}^{\frac{qp}{p-q}}(\omega)J(\omega)\,d\omega \int_0^{+\infty} \rho^{\frac{\eta qp}{p-q}+d-1}k^{\frac{p}{p-q}}(\rho,\omega)\,d\rho.$$

By (36) we have

$$\rho^{\nu r(p-q)-\eta q(p-r)} = \frac{(1-k(\rho,\omega))^{p-q}}{k^{r-q}(\rho,\omega)}\frac{\widetilde{\psi}^{q(p-r)}(\omega)}{\widetilde{\varphi}^{r(p-q)}(\omega)}. \tag{38}$$

Fixing ω, we pass to k

$$d\rho^{\frac{\eta qp}{p-q}+d} = \left(\frac{\widetilde{\psi}^{q(p-r)}(\omega)}{\widetilde{\varphi}^{r(p-q)}(\omega)}\right)^{\zeta} d\frac{(1-k)^{(p-q)\zeta}}{k^{(r-q)\zeta}}$$
$$= -\zeta\left(\frac{\widetilde{\psi}^{q(p-r)}(\omega)}{\widetilde{\varphi}^{r(p-q)}(\omega)}\right)^{\zeta}\frac{(1-k)^{(p-q)\zeta-1}}{k^{(r-q)\zeta+1}}(r-q+(p-r)k)\,dk,$$

where

$$\zeta = \frac{\eta qp + d(p-q)}{(p-q)(\nu r(p-q)-\eta q(p-r))} = \frac{q^*(1-\gamma)}{r(p-q)}.$$

Consequently,

$$\int_0^{+\infty}\rho^{\frac{\eta qp}{p-q}+d-1}k^{\frac{p}{p-q}}(\rho,\omega)\,d\rho$$
$$= \frac{p-q}{\eta qp + d(p-q)}\int_0^{+\infty}k^{\frac{p}{p-q}}(\rho,\omega)\,d\rho^{\frac{\eta qp}{p-q}+d}$$
$$= \frac{1}{|\nu r(p-q)-\eta q(p-r)|}\left(\frac{\widetilde{\psi}^{q(p-r)}(\omega)}{\widetilde{\varphi}^{r(p-q)}(\omega)}\right)^{\zeta}(K_1+K_2),$$

where

$$K_1 = (r-q)\int_0^1 k^{\widehat{p}}(1-k)^{\widehat{q}-1}\,dk = (r-q)B(\widehat{p}+1,\widehat{q}),$$

$$K_2 = (p-r)\int_0^1 k^{\widehat{p}+1}(1-k)^{\widehat{q}-1}\,dk = (p-r)B(\widehat{p}+2,\widehat{q})$$

$$= (p-r)\frac{\widehat{p}+1}{\widehat{p}+\widehat{q}+1}B(\widehat{p}+1,\widehat{q}),$$

$$\widehat{p} = \frac{qr(\nu-\eta)-d(r-q)}{\nu r(p-q)-\eta q(p-r)} = q^*\frac{\gamma}{p}, \quad \widehat{q} = \frac{\eta qp + d(p-q)}{\nu r(p-q)-\eta q(p-r)} = q^*\frac{1-\gamma}{r}.$$

Thus,

$$K_1+K_2 = p\frac{\nu r(p-q)-\eta q(p-r)}{\nu pr + d(p-r)}B(\widehat{p}+1,\widehat{q}) = \frac{pq}{q^*}B(\widehat{p}+1,\widehat{q})$$

$$= \frac{q\gamma}{q^*}\left(\frac{\gamma}{p}+\frac{1-\gamma}{r}\right)^{-1}B(\widehat{p},\widehat{q}).$$

The analogous calculations give

$$I_2 = \int_T |\psi(z)|^{\frac{qr}{p-q}}|\varphi(z)|^r k^{\frac{r}{p-q}}(z)\,d\mu(z)$$

$$= \int_\Omega \widetilde{\psi}^{\frac{qr}{p-q}}(\omega)\widetilde{\varphi}^r(\omega)J(\omega)\,d\omega \int_0^{+\infty} \rho^{\frac{\eta qr}{p-q}+\nu r+d-1}k^{\frac{r}{p-q}}(\rho,\omega)\,d\rho.$$

Fixing ω, we pass to k

$$d\rho^{\frac{\eta qr}{p-q}+\nu r+d} = \left(\frac{\widetilde{\psi}^{q(p-r)}(\omega)}{\widetilde{\varphi}^{r(p-q)}(\omega)}\right)^{\zeta_1} d\frac{(1-k)^{(p-q)\zeta_1}}{k^{(r-q)\zeta_1}}$$

$$= -\zeta_1\left(\frac{\widetilde{\psi}^{q(p-r)}(\omega)}{\widetilde{\varphi}^{r(p-q)}(\omega)}\right)^{\zeta_1} \frac{(1-k)^{(p-q)\zeta_1-1}}{k^{(r-q)\zeta_1+1}}(r-q+(p-r)k)\,dk,$$

where

$$\zeta_1 = \frac{\eta qr + (\nu r + d)(p-q)}{(p-q)(\nu r(p-q)-\eta q(p-r))} = \frac{q^*(1-\gamma)}{r(p-q)} + \frac{1}{p-q}.$$

We have

$$\int_0^{+\infty} \rho^{\frac{\eta qr}{p-q}+\nu r+d-1}k^{\frac{r}{p-q}}(\rho,\omega)\,d\rho$$

$$= \frac{p-q}{\eta qr + (\nu r+d)(p-q)}\int_0^{+\infty} k^{\frac{r}{p-q}}(\rho,\omega)\,d\rho^{\frac{\eta qr}{p-q}+\nu r+d}$$

$$= \frac{1}{|\nu r(p-q)-\eta q(p-r)|}\left(\frac{\widetilde{\psi}^{q(p-r)}(\omega)}{\widetilde{\varphi}^{r(p-q)}(\omega)}\right)^{\zeta_1}(L_1+L_2),$$

where

$$L_1 = (r-q)\int_0^1 k^{\widehat{p}-1}(1-k)^{\widehat{q}}\,dk = (r-q)B(\widehat{p},\widehat{q}+1),$$

$$L_2 = (p-r)\int_0^1 k^{\widehat{p}}(1-k)^{\widehat{q}}\,dk = (p-r)B(\widehat{p}+1,\widehat{q}+1)$$

$$= (p-r)\frac{\widehat{p}}{\widehat{p}+\widehat{q}+1}B(\widehat{p},\widehat{q}+1).$$

Thus,

$$L_1 + L_2 = r\frac{\nu r(p-q) - \eta q(p-r)}{\nu pr + d(p-r)}B(\widehat{p},\widehat{q}+1) = \frac{qr}{q^*}B(\widehat{p},\widehat{q}+1)$$

$$= \frac{q(1-\gamma)}{q^*}\left(\frac{\gamma}{p}+\frac{1-\gamma}{r}\right)^{-1}B(\widehat{p},\widehat{q}).$$

We obtain

$$I_1 = \frac{\gamma}{pr|\nu + d(1/r-1/p)|}\left(\frac{\gamma}{p}+\frac{1-\gamma}{r}\right)^{-1}B(\widehat{p},\widehat{q})I,$$

$$I_2 = \frac{1-\gamma}{pr|\nu + d(1/r-1/p)|}\left(\frac{\gamma}{p}+\frac{1-\gamma}{r}\right)^{-1}B(\widehat{p},\widehat{q})I.$$

It remains to apply Theorem 2. □

Note that for $d=1$ we have $I=1$ when $T=\mathbb{R}_+$ and $I=2$ when $T=\mathbb{R}$.

Assume that $|w(\cdot)|$, $|w_0(\cdot)|$, and $|w_1(\cdot)|$ are homogenous functions of degrees θ, θ_0, and θ_1, respectively. Define $\widetilde{w}(\cdot)$, $\widetilde{w}_0(\cdot)$, $\widetilde{w}_1(\cdot)$ by the analogy with (35).

From Theorem 2 (analogously to Corollary 3) we immediately obtain

Corollary 4 ([3][2])**.** *Suppose that $w(t), w_0(t), w_1(t) \neq 0$ for almost all $t \in T$, $1 \le q < p, r < \infty$, $\widetilde{\gamma} \in (0,1)$, where*

$$\widetilde{\gamma} = \frac{\widetilde{\theta}_1 - \widetilde{\theta}}{\widetilde{\theta}_1 - \widetilde{\theta}_0},$$

and $\widetilde{\theta}$, $\widetilde{\theta}_0$, and $\widetilde{\theta}_1$ are defined by (32)*. Moreover, assume that*

$$\widetilde{I} = \int_\Omega \frac{\widetilde{w}^{\widetilde{q}}(\omega)}{\widetilde{w}_0^{\widetilde{q}\widetilde{\gamma}}(\omega)\widetilde{w}_1^{\widetilde{q}(1-\widetilde{\gamma})}(\omega)}J(\omega)\,d\omega < \infty,$$

where

$$\frac{1}{\widetilde{q}} = \frac{1}{q} - \frac{\widetilde{\gamma}}{p} - \frac{1-\widetilde{\gamma}}{r}.$$

[2]The exact constant in [3] (formula (10)) was given with a misprint.

Then the exact inequality

$$\|w(\cdot)x(\cdot)\|_{L_q(T,\mu)} \le \widetilde{C}_1 \|w_0(\cdot)x(\cdot)\|^{\widetilde{\gamma}}_{L_p(T,\mu))} \|w_1(\cdot)x(\cdot)\|^{1-\widetilde{\gamma}}_{L_r(T,\mu)} \tag{39}$$

holds; here

$$\widetilde{C}_1 = \widetilde{\gamma}^{-\frac{\widetilde{\gamma}}{p}} (1-\widetilde{\gamma})^{-\frac{1-\widetilde{\gamma}}{r}} \left(\frac{B\left(\widetilde{q}\widetilde{\gamma}/p, \widetilde{q}(1-\widetilde{\gamma})/r\right) \widetilde{I}}{|\theta_1-\theta_0|(\widetilde{\gamma}r+(1-\widetilde{\gamma})p)} \right)^{1/\widetilde{q}}.$$

Put

$$w_0(t) = 1, \quad w_1(t) = t^{1-(\lambda+1)/p}, \quad w_2(t) = t^{1+(\mu-1)/q}.$$

From Corollary 4 we obtain

Corollary 5 ([5])**.** *Let $1 < p, q < \infty$ and $\lambda, \mu > 0$. Put*

$$\alpha = \frac{\mu}{p\mu + q\lambda}, \quad \beta = \frac{\lambda}{p\mu + q\lambda}.$$

Then the exact inequality

$$\|x(t)\|_{L_1(\mathbb{R}_+)} \le C \|t^{1-(\lambda+1)/p} x(t)\|^{p\alpha}_{L_p(\mathbb{R}_+)} \|t^{1+(\mu-1)/q} x(t)\|^{q\beta}_{L_q(\mathbb{R}_+)}$$

holds; here

$$C = \frac{1}{(p\alpha)^\alpha (q\beta)^\beta} \left(\frac{1}{\lambda+\mu} B\left(\frac{\alpha}{1-\alpha-\beta}, \frac{\beta}{1-\alpha-\beta} \right) \right)^{1-\alpha-\beta}.$$

Using Theorem $1'$ and calculations from the proofs of Theorems 2 and 3 we obtain

Theorem $3'$. *Let $1 < p, r < \infty$, $\widetilde{\varphi}(\omega), \widetilde{\psi}(\omega) \ne 0$ for almost all $\omega \in \Omega$ and γ, q^*, I, $k(\cdot)$, C_1, ξ_1 as above but for $q = 1$. Assume that $\gamma \in (0,1)$ and $I < \infty$. Then*

$$E_1(p, r) = C_1 \delta^\gamma.$$

Moreover, the method

$$\widehat{m}(y) = \int_T k\left(\xi_1^{\frac{1}{\nu + d(1/r - 1/p)}} t \right) \psi(t) y(t)\, d\mu(t)$$

is optimal recovery method.

4. Optimal recovery of functions from a noisy Fourier transform

Let S be the Schwartz space of rapidly decreasing C^∞-functions on $\mathbb{R}$, S' the corresponding space of distributions, and let $F\colon S' \to S'$ be the Fourier transform. We let $\mathcal{F}_p$ denote the space of distribution $x(\cdot)$ in S' for which

$$\|x(\cdot)\|_p = \left(\int_{\mathbb{R}} |Fx(t)|^p\, dt \right)^{1/p} < \infty, \quad 1 \le p < \infty.$$

We set

$$\mathcal{F}_p^n = \{\, x(\cdot) \in S' : \|x^{(n)}(\cdot)\|_p < \infty \,\},$$
$$F_p^n = \{\, x(\cdot) \in \mathcal{F}_p^n : \|x^{(n)}(\cdot)\|_p \le 1 \,\}.$$

Assume that the Fourier transform of a function $x(\cdot) \in F_r^n \cap \mathcal{F}_p$ is known on $\mathbb{R}$ to within $\delta > 0$ in the metric of $L_p(\mathbb{R})$. In other words, we know a function $y(\cdot) \in L_p(\mathbb{R})$ such that $\|Fx(\cdot) - y(\cdot)\|_{L_p(\mathbb{R})} \le \delta$. How should we best use this information to recover the lth derivative of the function in the metric $\mathcal{F}_q$, $0 \le l < n$? By recovery methods here we mean all possible mappings $m\colon L_p(\mathbb{R}) \to \mathcal{F}_q$. The error of a method is, by definition, the quantity

$$e_{p,q,r}(m) = \sup_{\substack{x(\cdot)\in F_r^n \cap \mathcal{F}_p,\ y(\cdot)\in L_p(\mathbb{R}) \\ \|Fx(\cdot)-y(\cdot)\|_{L_p(\Delta_\sigma)} \le \delta}} \|x^{(l)}(\cdot) - m(y)(\cdot)\|_q.$$

The optimal recovery error is defined as follows:

$$E_{p,q,r} = \inf_{m\colon L_p(\mathbb{R}) \to \mathcal{F}_q} e_{p,q,r}(m).$$

A method on which this lower bound is attained is called optimal.

It is readily checked that this problem is a special case of the general problem (1) with $T = T_0 = \mathbb{R}$, $\psi(t) = (it)^l$, $\varphi(t) = (it)^n$.

The cases 1) $1 \le q = r < p < \infty$, 2) $1 \le q = p < r < \infty$, 3) $1 \le q = p = r < \infty$, and 4) $1 \le q < p = r < \infty$ were studied in [14].

For the case $1 \le q < p, r < \infty$ we can apply Theorem 3. In this case

$$\frac{k^{\frac{1}{p-q}}(t)}{(1-k(t))^{\frac{1}{r-q}}} = |t|^{\frac{lq(p-r)-nr(p-q)}{(p-q)(r-q)}}, \quad \gamma = \frac{n-l-1/q+1/r}{n+1/r-1/p},$$

and $I = 2$. It is easy to verify that if $n > l + 1/q - 1/r$, then $\gamma \in (0,1)$. Thus, it follows by Theorem 3

Theorem 4. *Let $1 \le q < p, r < \infty$ and $n > l + 1/q - 1/r$. Then*

$$E_{p,q,r} = C_1 \delta^\gamma, \tag{40}$$

where

$$C_1 = \gamma^{-\frac{\gamma}{p}} (1-\gamma)^{-\frac{1-\gamma}{r}} \left(\frac{2B\left(q^*\gamma/p, q^*(1-\gamma)/r\right)}{(n+1/r-1/p)(\gamma r + (1-\gamma)p)} \right)^{1/q^*}$$

and q^* is defined by (37)*. Moreover, the method $\widehat{m}(y)(\cdot) = F^{-1}Y_y(\cdot)$ is optimal, where*

$$Y_y(t) = (it)^l k\left(\xi_1^{\frac{1}{n+1/r-1/p}} t\right) y(t), \quad \xi_1 = \delta \left(\gamma^{q-r}(1-\gamma)^{p-q} C_1^{p-r}\right)^{\frac{q^*}{pqr}}.$$

Note that case 4) immediately follows from Theorem 4 for $p = r$. In cases 1)–3) the optimal recovery error coincides with the limits $\lim_{r\to q} E_{p,q,r}$, $\lim_{p\to q} E_{p,q,r}$, $\lim_{p\to q} E_{p,q,p}$, respectively, where $E_{p,q,r}$ is given by (40).

5. Optimal recovery of derivatives and generalized Carlson-Levin-Taikov inequalities

For functions $x(\cdot) \in L_2(\mathbb{R})$ whose $(n-1)$st derivative is locally absolutely continuous and $0 \le k \le n-1$, L. V. Taikov [16] obtained exact inequality

$$|x^{(k)}(0)| \le K \|x(\cdot)\|_{L_2(\mathbb{R})}^{\frac{2n-2k-1}{2n}} \|x^{(n)}(\cdot)\|_{L_2(\mathbb{R})}^{\frac{2k+1}{2n}},$$

where

$$K = \left(\frac{2k+1}{2n-2k-1}\right)^{\frac{2n-2k-1}{4n}} \left((2k+1)\sin\frac{2k+1}{2n}\pi\right)^{-1/2}.$$

Passing to the Fourier transform we have the following equivalent inequality

$$\left|\frac{1}{2\pi}\int_{\mathbb{R}} t^k Fx(t)\,dt\right| \le K\left(\frac{1}{2\pi}\int_{\mathbb{R}} |Fx(t)|^2\,dt\right)^{\frac{2n-2k-1}{4n}} \times \left(\frac{1}{2\pi}\int_{\mathbb{R}} t^{2n}|Fx(t)|^2\,dt\right)^{\frac{2k+1}{4n}}.$$

Set $g(t) = t^k Fx(t)$. Then we obtain the following inequality

$$\left|\int_{\mathbb{R}} g(t)\,dt\right| \le K\sqrt{2\pi}\left(\int_{\mathbb{R}} t^{-2k}|g(t)|^2\,dt\right)^{\frac{2n-2k-1}{4n}} \times \left(\int_{\mathbb{R}} t^{2(n-k)}|g(t)|^2\,dt\right)^{\frac{2k+1}{4n}}.$$

Put $p = q = 2$, $\lambda = 2k+1$, and $\mu = 2n-2k-1$. Then by Corollary 4 we have

$$\int_0^\infty |g(t)|\,dt \le C\left(\int_0^\infty t^{-2k}|g(t)|^2\,dt\right)^{\frac{2n-2k-1}{4n}} \times \left(\int_0^\infty t^{2(n-k)}|g(t)|^2\,dt\right)^{\frac{2k+1}{4n}},$$

where

$$C = \left(\frac{2k+1}{2n-2k-1}\right)^{\frac{2n-2k-1}{4n}} (2k+1)^{-1/2} B^{1/2}\left(\frac{2n-2k-1}{2n}, \frac{2k+1}{2n}\right).$$

Since

$$B\left(1-\frac{2k+1}{2n}, \frac{2k+1}{2n}\right) = \frac{\pi}{\sin\dfrac{2k+1}{2n}\pi}$$

we have

$$C = \sqrt{\pi}\left(\frac{2k+1}{2n-2k-1}\right)^{\frac{2n-2k-1}{4n}} \left((2k+1)\sin\frac{2k+1}{2n}\pi\right)^{-1/2}.$$

From the inequality

$$a_1b_1 + a_2b_2 \le 2^{1-s-t}(a_1^{1/r} + a_2^{1/r})^r(b_1^{1/s} + b_2^{1/s})^s$$

it follows that

$$\begin{aligned}\int_{\mathbb{R}} |g(t)|\,dt &= \int_{-\infty}^0 |g(t)|\,dt + \int_0^\infty |g(t)|\,dt\\ &\le C\biggl(\int_{-\infty}^0 t^{-2k}|g(t)|^2\,dt\biggr)^{\frac{2n-2k-1}{4n}}\biggl(\int_{-\infty}^0 t^{2(n-k)}|g(t)|^2\,dt\biggr)^{\frac{2k+1}{4n}}\\ &+ C\biggl(\int_0^\infty t^{-2k}|g(t)|^2\,dt\biggr)^{\frac{2n-2k-1}{4n}}\biggl(\int_0^\infty t^{2(n-k)}|g(t)|^2\,dt\biggr)^{\frac{2k+1}{4n}}\\ &\le \sqrt{2}C\biggl(\int_{\mathbb{R}} t^{-2k}|g(t)|^2\,dt\biggr)^{\frac{2n-2k-1}{4n}}\biggl(\int_{\mathbb{R}} t^{2(n-k)}|g(t)|^2\,dt\biggr)^{\frac{2k+1}{4n}}.\end{aligned}$$

Thus Taikov's inequality follows from Levin's inequality.

This inequality is closely connected with the problem of optimal recovery of derivatives from inaccurate information about the Fourier transform (see [9]). We consider such problem in multidimensional case.

Consider linear operators $D_1\colon L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)\cap C(\mathbb{R}^d)$ and $D_2\colon L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$ (D_1 and D_2 are not necessary differentiation operators). Put

$$W = \{\, x(\cdot) \in L_2(\mathbb{R}^d) : \|D_2x(\cdot)\|_{L_2(\mathbb{R}^d)} \le 1 \,\}.$$

We consider the problem of optimal recovery of $D_1x(\tau)$, $\tau \in \mathbb{R}^d$, on the class W from the information about $x(\cdot)$, given inaccurately in $L_2(\mathbb{R}^d)$-metric.

As recovery methods we consider all possible mappings $m\colon L_2(\mathbb{R}^d) \to \mathbb{C}$ or $\mathbb{R}$. The error of a method m is defined as

$$e(m) = \sup_{\substack{x(\cdot)\in W,\ y(\cdot)\in L_2(\mathbb{R}^d)\\ \|x(\cdot)-y(\cdot)\|_{L_2(\mathbb{R}^d)}\le\delta}} |D_1x(\tau) - m(y)|.$$

The quantity

$$E = \inf_{m\colon L_2(\mathbb{R}^d)\to\mathbb{C}(\mathbb{R})} e(m) \tag{41}$$

is known as the optimal recovery error, and a method on which this infimum is attained is called optimal.

For the case when $d = 1$, $D_1x(\cdot) = x^{(k)}(\cdot)$, and $D_2x(\cdot) = x^{(n)}(\cdot)$, $0 \le k < n$, similar problems were considered in [9].

Let $d_1(t)$ and $d_2(\cdot)$ be measurable functions on R^d. Put

$$X = \{\, x(\cdot) \in L_2(\mathbb{R}^d) : d_2(\cdot)Fx(\cdot) \in L_2(\mathbb{R}^d) \,\}.$$

We define the operator D_2 as follows

$$D_2x(\cdot) = F^{-1}(d_2(\cdot)Fx(\cdot))(\cdot).$$

Assume that $d_1(\cdot)Fx(\cdot) \in L_2(\mathbb{R}^d)$ for all $x(\cdot) \in X$ and the operator D_1 which is defined by the equality

$$D_1x(\cdot) = F^{-1}(d_1(\cdot)Fx(\cdot))(\cdot)$$

maps X to $L_2(\mathbb{R}^d) \cap C(\mathbb{R}^d)$.

Let $|d_1(\cdot)|$ and $|d_2(\cdot)|$ be homogenous functions of degrees k, n, respectively (k and n are not necessarily integer), $d_j(t) \neq 0$, $j = 1, 2$, for almost all $t \in \mathbb{R}^d$. Put

$$\widetilde{d}_1(\omega) = \rho^{-k}|d_1(\rho\cos\omega_1, \ldots, \rho\sin\omega_1\sin\omega_2\ldots\sin\omega_{d-2}\sin\omega_{d-1})|,$$
$$\widetilde{d}_2(\omega) = \rho^{-n}|d_2(\rho\cos\omega_1, \ldots, \rho\sin\omega_1\sin\omega_2\ldots\sin\omega_{d-2}\sin\omega_{d-1})|.$$

By Plancherel's theorem we have

$$W = \left\{ x(\cdot) \in L_2(\mathbb{R}^d) : \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |d_2(t)Fx(t)|^2\,dt \le 1 \right\},$$
$$\|x(\cdot) - y(\cdot)\|_{L_2(\mathbb{R}^d)} = \frac{1}{(2\pi)^{d/2}}\|Fx(\cdot) - Fy(\cdot)\|_{L_2(\mathbb{R}^d)}.$$

Moreover,

$$D_1x(\tau) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} d_1(t)Fx(t)e^{i\langle\tau,t\rangle}\,dt,$$

where $\langle\tau, t\rangle = \tau_1t_1 + \ldots + \tau_dt_d$. Thus we obtain problem (23) with $p = r = 2$, $\delta_1 = \delta(2\pi)^{d/2}$,

$$\psi(t) = \frac{1}{(2\pi)^d}d_1(t)e^{i\langle\tau,t\rangle}, \quad \varphi(t) = \frac{1}{(2\pi)^{d/2}}d_2(t).$$

By Theorem $3'$ we have

Theorem 5. *Let $k \ge 0$ and $n > k + d/2$. Assume that*

$$I = \int_{\Pi_{d-1}} \frac{\widetilde{d}_1^{\,2}(\omega)}{\widetilde{d}_2^{\,\frac{2k+d}{n}}(\omega)}J(\omega)\,d\omega < \infty, \quad \Pi_{d-1} = [0,\pi]^{d-2} \times [0, 2\pi].$$

Then

$$E = \frac{(\pi I)^{1/2}}{(2\pi)^{d/2}}K_d(k,n)\delta^{\frac{2n-2k-d}{2n}},$$

where

$$K_d(k,n) = \left(\frac{2k+d}{2n-2k-d}\right)^{\frac{2n-2k-d}{4n}}\left((2k+d)\sin\frac{2k+d}{2n}\pi\right)^{-1/2}.$$

Moreover, the method

$$\widehat{m}(y) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} d_1(t)\left(1 + \frac{\delta^2(2k+d)}{(2\pi)^d(2n-2k-d)}\right)^{-1} y(t)e^{i\langle\tau,t\rangle}\,dt$$

is optimal recovery method.

By this theorem analogously to (31) we obtain the exact inequality

$$|D_1x(\tau)| \le \frac{(\pi I)^{1/2}}{(2\pi)^{d/2}} K_d(k,n) \|x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\frac{2n-2k-d}{2n}} \|D_2x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\frac{2k+d}{2n}}$$

or

$$\|D_1x(\cdot)\|_{L_\infty(\mathbb{R}^d)} \le \frac{(\pi I)^{1/2}}{(2\pi)^{d/2}} K_d(k,n) \|x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\frac{2n-2k-d}{2n}} \|D_2x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\frac{2k+d}{2n}}. \tag{42}$$

Now we consider some examples. Define the operator $(-\Delta)^{n/2}$, $n \ge 0$, as follows

$$(-\Delta)^{n/2}x(\cdot) = F^{-1}(|t|^n Fx(t))(\cdot).$$

Put $d_1(t) = |t|^k$ and $d_2(t) = |t|^n$. Then problem (41) is the problem of optimal recovery of $(-\Delta)^{k/2}x(\tau)$ on the class

$$W = \{\, x(\cdot) \in L_2(\mathbb{R}^d) : \|(-\Delta)^{n/2}x(\cdot)\|_{L_2(\mathbb{R}^d)} \le 1 \,\}$$

by the inaccurate information about $x(\cdot)$.

By Theorem 5 we obtain

Corollary 6. *Let $n > k + d/2$. Then*

$$E = C_d(k,n)\delta^{\frac{2n-2k-d}{2n}}, \quad C_d(k,n) = \frac{K_d(k,n)}{(2^{d-1}\pi^{d/2-1}\Gamma(d/2))^{1/2}},$$

and the method

$$\widehat{m}(y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |t|^k \left(1 + \frac{\delta^2(2k+d)}{(2\pi)^d(2n-2k-d)}\right)^{-1} y(t)e^{i\langle \tau, t\rangle}\, dt$$

is optimal.

By (42) we get the exact inequality

$$\|(-\Delta)^{k/2}x(\cdot)\|_{L_\infty(\mathbb{R}^d)} \le C_d(k,n)\|x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\frac{2n-2k-d}{2n}} \|(-\Delta)^{n/2}x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\frac{2k+d}{2n}}.$$

Consider one more example. Let $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}_+^d$. We define D^α (the derivative of order α) as follows:

$$D^\alpha x(\cdot) = F^{-1}((it)^\alpha Fx(t))(\cdot),$$

where $(it)^\alpha = (it_1)^{\alpha_1} \cdots (it_d)^{\alpha_d}$. Let $D_1 = D^\alpha$ and $D_2 = (-\Delta)^{n/2}$. Then (41) is the problem of optimal recovery of $D^\alpha x(\tau)$ on the class W by the inaccurate information about $x(\cdot)$.

From the well-known Dirichlet formula we have

$$\int_{\substack{x_1\ge 0,\ldots,x_d\ge 0\\ x_1^2+\ldots+x_d^2\le 1}} x_1^{p_1-1}\ldots x_d^{p_d-1}\, dx_1\ldots dx_d = \frac{\Gamma(p_1/2)\ldots\Gamma(p_d/2)}{2^d\Gamma(p_1/2+\ldots+p_d/2+1)},$$

$p_1, \ldots, p_d > 0$. Using this formula and passing to the polar transformation we obtain

$$I(p_1, \ldots, p_d) = \int_{\Pi_{d-1}} \Phi(\omega, p_1, \ldots, p_d) J(\omega)\, d\omega = 2\frac{\Gamma(p_1/2)\ldots\Gamma(p_d/2)}{\Gamma(p_1/2 + \ldots + p_d/2)},$$

where

$$\begin{aligned}\Phi(\omega, p_1, \ldots, p_d) = |\cos\omega_1|^{p_1-1} |\sin\omega_1 \cos\omega_2|^{p_2-1} \times \ldots \\ \times |\sin\omega_1 \sin\omega_2 \ldots \sin\omega_{d-2} \cos\omega_{d-1}|^{p_{d-1}-1} \\ \times |\sin\omega_1 \sin\omega_2 \ldots \sin\omega_{d-2} \sin\omega_{d-1}|^{p_d-1}.\end{aligned}$$

Thus for $d_1(t) = (it)^\alpha$ and $d_2(t) = |t|^n$ we have

$$I = I(2\alpha_1 + 1, \ldots, 2\alpha_d + 1) = 2\frac{\Gamma(\alpha_1 + 1/2)\ldots\Gamma(\alpha_d + 1/2)}{\Gamma(|\alpha| + d/2)},$$

where $|\alpha| = \alpha_1 + \ldots \alpha_d$.

Corollary 7. *Let $n > |\alpha| + d/2$. Then*

$$E = C_{d,\alpha}(n)\delta^{\frac{2n-2|\alpha|-d}{2n}},$$

where

$$C_{d,\alpha}(n) = \frac{K_d(|\alpha|, n)}{(2\pi)^{(d-1)/2}} \left(\frac{\Gamma(\alpha_1 + 1/2)\ldots\Gamma(\alpha_d + 1/2)}{\Gamma(|\alpha| + d/2)}\right)^{1/2},$$

and the method

$$\widehat{m}(y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (it)^\alpha \left(1 + \frac{\delta^2(2|\alpha| + d)}{(2\pi)^d(2n - 2|\alpha| - d)}\right)^{-1} y(t)e^{i\langle\tau,t\rangle}\, dt$$

is optimal.

The exact inequality in this case has the form:

$$\|D^\alpha x(\cdot)\|_{L_\infty(\mathbb{R}^d)} \le C_{d,\alpha}(n)\|x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\frac{2n-2|\alpha|-d}{2n}} \|(-\Delta)^{n/2}x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\frac{2|\alpha|+d}{2n}}.$$

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