Recovery of Derivatives for Functions Defined on the Semiaxis

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Abstract

The paper is concerned with the recovery problem of derivatives at the origin from noisy information about functions defined on the semiaxis for the Sobolev class. The problem of S. B. Stechkin about approximation of derivatives by bounded linear functionals is also studied.

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1. Setting of Problems

Let $\mathcal{W}_2^n(\mathbb{R}_+)$, $n \in \mathbb{N}$, be the Sobolev space of functions $x(\cdot) \in L_2(\mathbb{R}_+)$ such that the $(n-1)$-st derivative is locally absolutely continuous and $x^{(n)}(\cdot) \in L_2(\mathbb{R}_+)$. Set

$$W_2^n(\mathbb{R}_+) = \{\, x(\cdot) \in \mathcal{W}_2^n(\mathbb{R}_+) : \|x^{(n)}(\cdot)\|_{L_2(\mathbb{R}_+)} \le 1 \,\}.$$

We consider the problem of recovery of $x^{(k)}(0)$, $0 \le k < n$, on the class $W_2^n(\mathbb{R}_+)$ by inaccurate information about $x(\cdot)$. We assume that for every function $x(\cdot) \in W_2^n(\mathbb{R}_+)$ we know $y(\cdot) \in L_2(\mathbb{R}_+)$ such that

$$\|x(\cdot) - y(\cdot)\|_{L_2(\mathbb{R}_+)} \le \delta, \quad \delta > 0.$$

For a given $y(\cdot)$ we have to give an approximate value of $x^{(k)}(0)$.

As recovery methods we consider all possible mappings $m\colon L_2(\mathbb{R}_+) \to \mathbb{R}$. The error of a method m is defined as follows

$$e_k(W_2^n(\mathbb{R}_+), \delta, m) = \sup_{\substack{x(\cdot) \in W_2^n(\mathbb{R}_+),\ y(\cdot) \in L_2(\mathbb{R}_+) \\ \|x(\cdot) - y(\cdot)\|_{L_2(\mathbb{R}_+)} \le \delta}} |x^{(k)}(0) - m(y(\cdot))|.$$

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The quantity

$$E_k(W_2^n(\mathbb{R}_+),\delta)=\inf_{m\colon L_2(\mathbb{R}_+)\to\mathbb{R}} e_k(W_2^n(\mathbb{R}_+),\delta,m)$$

is known as the optimal recovery error and a method for which this infimum is attained is called optimal.

We also study the problem of best approximation of $x^{(k)}(0)$ on the class $W_2^n(\mathbb{R}_+)$ by inaccurate information about $x(\cdot)$ by linear continuous functionals on $L_2(\mathbb{R}_+)$ with the norm not greater than some fixed positive number N (this problem is known as Stechkin's problem). It is in finding the value

$$S_k(W_2^n(\mathbb{R}_+),N)=\inf_{\substack{y^*\in L_2(\mathbb{R}_+)\\ \|y^*\|\le N}}\ \sup_{x(\cdot)\in L_2(\mathbb{R}_+)} |x^{(k)}(0)-\langle y^*,x(\cdot)\rangle|,$$

and also a functional delivering the lower bound which is called extremal.

The solutions of formulated problems are closely connected with the problems of exact constants in Kolmogorov-type inequalities for derivatives. For the semiaxis the corresponding results which we essentially use here where obtained in [2] and [3]. For $\mathbb{R}$ the analogous problems of recovery and approximation by bounded linear functionals were considered in [5]. The range of problems connected with Stechkin's problem was elucidated in the survey paper [1].

2. Main Results

It may be obtained from [3] that there exists a function $\widehat{x}(\cdot)\in\mathcal{W}^{2n}(\mathbb{R}_+)$ such that for all $x(\cdot)\in\mathcal{W}_2^n(\mathbb{R}_+)$ the following equality

$$x^{(k)}(0)=\int_{\mathbb{R}_+} x(t)\overline{\widehat{x}(t)}\,dt+\int_{\mathbb{R}_+} x^{(n)}(t)\overline{\widehat{x}^{(n)}(t)}\,dt \tag{1}$$

holds. We give the explicit form of $\widehat{x}(\cdot)$ and prove the corresponding result for completeness.

Put

$$\lambda_j=e^{(n+2j-1)\frac{i\pi}{2n}}\quad j=1,\dots,n,\quad A=\begin{pmatrix}\lambda_1^n & \dots & \lambda_n^n\\ \dots & \dots & \dots\\ \lambda_1^{2n-1} & \dots & \lambda_n^{2n-1}\end{pmatrix},$$

and A_{sj} is the cofactor of λ_j^{n+s-1}.

Lemma 1. *Set*

$$\widehat{x}(t)=\frac{(-1)^{n-k}}{|A|}\sum_{j=1}^n A_{n-k,j}e^{\lambda_j t}.$$

Then for all $x(\cdot)\in\mathcal{W}_2^n(\mathbb{R}_+)$ (1) holds, moreover,

$$\widehat{x}^{(k)}(0)=A_{nk}^2,$$

where

$$A_{nk}=\frac{1}{\sin^{1/2}((2k+1)\alpha)}\prod_{j=1}^{k}\cot j\alpha,\quad \alpha=\frac{\pi}{2n}.$$

Proof. Since $\lambda_j^{2n}=(-1)^{n-1}$ for all $j=1,\dots,n$ we have

$$\widehat{x}(t)+(-1)^n\widehat{x}^{(2n)}(t)=0.$$

In view of the fact that $\operatorname{Re}\lambda_j<0$ for all $j=1,\dots,n$, $\widehat{x}(\cdot)\in\mathcal{W}^{2n}(\mathbb{R}_+)$. Thus, for every $x(\cdot)\in\mathcal{W}_2^n(\mathbb{R}_+)$ we have

$$\int_{\mathbb{R}_+}x(t)\overline{\widehat{x}(t)}\,dt+(-1)^n\int_{\mathbb{R}_+}x(t)\overline{\widehat{x}^{(2n)}(t)}\,dt=0.$$

Using integration by parts we obtain

$$\int_{\mathbb{R}_+}x(t)\overline{\widehat{x}(t)}\,dt+(-1)^n\sum_{p=1}^{n}(-1)^px^{(p-1)}(0)\overline{\widehat{x}^{(2n-p)}(0)}+\int_{\mathbb{R}_+}x^{(n)}(t)\overline{\widehat{x}^{(n)}(t)}\,dt=0.\quad (2)$$

For all $s,p=1,\dots,n$ we have

$$\sum A_{sj}\lambda_j^{n+p-1}=\delta_{ps}|A|.$$

Consequently,

$$\widehat{x}^{(2n-p)}(0)=\frac{(-1)^{n-k}}{|A|}\sum_{j=1}^{n}A_{n-k,j}\lambda_j^{2n-p}=(-1)^{n-k}\delta_{n-p+1,n-k}.\quad (3)$$

Substituting (3) in (2) we obtain (1).

Now let us calculate $\widehat{x}^{(k)}(0)$. We have

$$\widehat{x}^{(k)}(0)=\frac{(-1)^{n-k}}{|A|}\sum_{j=1}^{n}A_{n-k,j}\lambda_j^{k}=(-1)^{n-k}\frac{|A_k|}{|A|},$$

where

$$A_k=\begin{pmatrix}\lambda_1^n & \dots & \lambda_n^n\\ \dots & \dots & \dots\\ \lambda_1^{2n-k-2} & \dots & \lambda_n^{2n-k-2}\\ \lambda_1^{k} & \dots & \lambda_n^{k}\\ \lambda_1^{2n-k} & \dots & \lambda_n^{2n-k}\\ \dots & \dots & \dots\\ \lambda_1^{2n-1} & \dots & \lambda_n^{2n-1}\end{pmatrix}.$$

Since

$$\lambda_j^m=\sigma_m\mu_m^{j-1},$$

where

$$|\sigma_m| = 1, \quad \mu_m = e^{i\frac{\pi m}{n}},$$

matrixes A and A_k may be rewritten in the forms

$$A = \begin{pmatrix} \sigma_n & \dots & \sigma_n \mu_n^{n-1} \\ \dots & \dots & \dots \\ \sigma_{2n-1} & \dots & \sigma_{2n-1}\mu_{2n-1}^{n-1} \end{pmatrix}$$

$$A_k = \begin{pmatrix} \sigma_n & \dots & \sigma_n \mu_n^{n-1} \\ \dots & \dots & \dots \\ \sigma_{2n-k-2} & \dots & \sigma_{2n-k-2}\mu_{2n-k-2}^{n-1} \\ \sigma_k & \dots & \sigma_k \mu_k^{n-1} \\ \sigma_{2n-k} & \dots & \sigma_{2n-k}\mu_{2n-k}^{n-1} \\ \dots & \dots & \dots \\ \sigma_{2n-1} & \dots & \sigma_{2n-1}\mu_{2n-1}^{n-1} \end{pmatrix}.$$

If we put $x(\cdot) = \widehat{x}(\cdot)$ in (1), then we obtain

$$\widehat{x}^{(k)}(0) = \|\widehat{x}(\cdot)\|^2_{L_2(\mathbb{R}_+)} + \|\widehat{x}^{(n)}(\cdot)\|^2_{L_2(\mathbb{R}_+)}. \tag{4}$$

Thus, $\widehat{x}^{(k)}(0) > 0$. Using this fact and the formula for the Vandermonde determinant we have

$$\widehat{x}^{(k)}(0) = \prod_{\substack{j=1 \\ j\neq n-k}}^{n} \frac{|\mu_k - \mu_{n+j-1}|}{|\mu_{2n-k-1} - \mu_{n+j-1}|}.$$

Since

$$|\mu_p - \mu_s| = 2|\sin(p-s)\alpha|, \quad \alpha = \frac{\pi}{2n},$$

we obtain

$$\widehat{x}^{(k)}(0) = \prod_{\substack{j=1 \\ j\neq n-k}}^{n} \frac{|\sin(n+j-k-1)\alpha|}{|\sin(n+j+k)\alpha|}$$

$$= \frac{1}{\sin((2k+1)\alpha)} \frac{\prod_{j=k+1}^{k+n} \sin j\alpha}{\prod_{j=1}^{k} \sin j\alpha \prod_{j=1}^{n-k-1} \sin j\alpha}$$

$$= \frac{1}{\sin((2k+1)\alpha)} \frac{\prod_{j=k+1}^{n-1} \sin j\alpha \prod_{j=n+1}^{n+k} \sin j\alpha}{\prod_{j=1}^{k} \sin j\alpha \prod_{j=1}^{n-k-1} \sin j\alpha} =$$

$$\frac{1}{\sin((2k+1)\alpha)} \prod_{j=1}^{k} \cot j\alpha \frac{\prod_{j=k+1}^{n-1} \sin j\alpha}{\prod_{j=1}^{n-k-1} \sin j\alpha} =$$

$$\frac{1}{\sin((2k+1)\alpha)} \prod_{j=1}^{k} \cot j\alpha \prod_{j=1}^{n-k-1} \cot j\alpha = \frac{1}{\sin((2k+1)\alpha)} \prod_{j=1}^{k} \cot^2 j\alpha.$$

$\square$

Theorem 1. *The following equality*

$$E_k(W_2^n(\mathbb{R}_+),\delta) = A_{nk}\left(\frac{2n}{2n-2k-1}\right)^{\frac{2n-2k-1}{4n}}\left(\frac{2n}{2k+1}\right)^{\frac{2k+1}{4n}}\delta^{\frac{2n-2k-1}{2n}}$$

holds. Moreover,

$$\widehat{m}(y) = \beta^{k+1}\int_{\mathbb{R}_+} y(t)\overline{\widehat{x}(\beta t)}\,dt, \tag{5}$$

where

$$\beta = \left(\frac{2n-2k-1}{2k+1}\right)^{\frac{1}{2n}}\delta^{-\frac{1}{n}},$$

is optimal method of recovery.

Proof. From (1) by the Cauchy-Schwartz inequality we obtain

$$|x^{(k)}(0)| \le \left(\|\widehat{x}(\cdot)\|^2_{L_2(\mathbb{R}_+)} + \|\widehat{x}^{(n)}(\cdot)\|^2_{L_2(\mathbb{R}_+)}\right)^{1/2} \times \left(\|x(\cdot)\|^2_{L_2(\mathbb{R}_+)} + \|x^{(n)}(\cdot)\|^2_{L_2(\mathbb{R}_+)}\right)^{1/2}.$$

Taking into account (4) we have

$$|x^{(k)}(0)| \le A_{nk}\left(\|x(\cdot)\|^2_{L_2(\mathbb{R}_+)} + \|x^{(n)}(\cdot)\|^2_{L_2(\mathbb{R}_+)}\right)^{1/2}. \tag{6}$$

Put $y(t) = \widehat{x}(bt)$, $b > 0$. Then

$$\|y(\cdot)\|^2_{L_2(\mathbb{R}_+)} = \frac{1}{b}\|\widehat{x}(\cdot)\|^2_{L_2(\mathbb{R}_+)}, \quad \|y^{(n)}(\cdot)\|^2_{L_2(\mathbb{R}_+)} = b^{2n-1}\|\widehat{x}^{(n)}(\cdot)\|^2_{L_2(\mathbb{R}_+)}.$$

Substituting $y(\cdot)$ into (6) we obtain that for all $b > 0$ the inequality

$$b^k A_{nk}^2 \le A_{nk}\left(\frac{1}{b}\|\widehat{x}(\cdot)\|^2_{L_2(\mathbb{R}_+)} + b^{2n-1}\|\widehat{x}^{(n)}(\cdot)\|^2_{L_2(\mathbb{R}_+)}\right)^{1/2}$$

holds. In view of (4) we get that for all $b > 0$ the inequality $f(b) \ge 0$ is fulfilled, where

$$f(b) = b^{2n}\|\widehat{x}^{(n)}(\cdot)\|^2_{L_2(\mathbb{R}_+)} - A_{nk}^2 b^{2k+1} + A_{nk}^2 - \|\widehat{x}^{(n)}(\cdot)\|^2_{L_2(\mathbb{R}_+)}.$$

It is easily seen that $f(\cdot)$ has the unique minimum on $\mathbb{R}_+$

$$b_0 = \left(\frac{(2k+1)A_{nk}^2}{2n\|\widehat{x}^{(n)}(\cdot)\|^2_{L_2(\mathbb{R}_+)}}\right)^{\frac{1}{2n-2k-1}}.$$

On the other hand, $f(1) = 0$. Consequently, $b_0 = 1$. Thus,

$$\|\widehat{x}^{(n)}(\cdot)\|_{L_2(\mathbb{R}_+)} = A_{nk}\sqrt{\frac{2k+1}{2n}}.$$

It follows from (4) that

$$\|\widehat{x}(\cdot)\|_{L_2(\mathbb{R}_+)} = A_{nk}\sqrt{\frac{2n-2k-1}{2n}}.$$

Put $\widehat{x}_1(t) = \alpha\widehat{x}(\beta t)$, $\alpha, \beta > 0$. Choose α and β such that $\|\widehat{x}_1(\cdot)\|_{L_2(\mathbb{R}_+)} = \delta$ and $\|\widehat{x}_1^{(n)}(\cdot)\|_{L_2(\mathbb{R}_+)} = 1$. We have

$$\alpha^2\beta^{-1}\|\widehat{x}(\cdot)\|^2_{L_2(\mathbb{R}_+)} = \delta^2, \quad \alpha^2\beta^{2n-1}\|\widehat{x}^{(n)}(\cdot)\|^2_{L_2(\mathbb{R}_+)} = 1.$$

Hence,

$$\begin{aligned}
\alpha &= A_{nk}^{-1}\sqrt{\frac{2n}{2n-2k-1}}\left(\frac{2n-2k-1}{2k+1}\right)^{\frac{1}{4n}}\delta^{1-\frac{1}{2n}},\\
\beta &= \left(\frac{2n-2k-1}{2k+1}\right)^{\frac{1}{2n}}\delta^{-\frac{1}{n}}.
\end{aligned}$$

Substituting $\widehat{x}(t) = \alpha^{-1}\widehat{x}_1(t/\beta)$ in (1) we obtain

$$x^{(k)}(0) = \alpha^{-1}\int_{\mathbb{R}_+} x(t)\overline{\widehat{x}_1(t/\beta)}\,dt + \alpha^{-1}\beta^{-n}\int_{\mathbb{R}_+} x^{(n)}(t)\overline{\widehat{x}_1^{(n)}(t/\beta)}\,dt.$$

We change variables $t = \beta s$ and put $z(s) = x(\beta s)$. Then we have that for all $z(\cdot) \in \mathcal{W}_2^n(\mathbb{R}_+)$ the equality

$$z^{(k)}(0) = \lambda_1\int_{\mathbb{R}_+} z(s)\overline{\widehat{x}_1(s)}\,ds + \lambda_2\int_{\mathbb{R}_+} z^{(n)}(s)\overline{\widehat{x}_1^{(n)}(s)}\,ds \tag{7}$$

holds with

$$\lambda_1 = \frac{\beta^{k+1}}{\alpha}, \quad \lambda_2 = \frac{1}{\alpha\beta^{2n-k-1}}.$$

It follows from general results about optimal recovery of linear functionals (see, for example, [4]) that

$$E_k(W_2^n(\mathbb{R}_+), \delta) = \sup_{\substack{z(\cdot)\in W_2^n(\mathbb{R}_+)\\ \|z(\cdot)\|_{L_2(\mathbb{R}_+)}\le\delta}} |z^{(k)}(0)|. \tag{8}$$

From (7) by the Cauchy-Schwartz inequality we obtain

$$E_k(W_2^n(\mathbb{R}_+), \delta) \le \lambda_1\delta^2 + \lambda_2. \tag{9}$$

Let us estimate the error of method (5), which may be written in the following way

$$\widehat{m}(y) = \lambda_1\int_{\mathbb{R}_+} y(t)\overline{\widehat{x}_1(t)}\,dt.$$

Suppose that $z(\cdot)\in W_2^n(\mathbb{R}_+)$ and $\|z(\cdot)-y(\cdot)\|_{L_2(\mathbb{R}_+)}\le\delta$. Taking into account (7) we have

$$\begin{aligned}
&|z^{(k)}(0)-\widehat{m}(y)|\\
&\qquad=\left|z^{(k)}(0)-\lambda_1\int_{\mathbb{R}_+}z(t)\overline{\widehat{x}_1(t)}\,dt+\lambda_1\int_{\mathbb{R}_+}(z(t)-y(t)\overline{\widehat{x}_1(t)}\,dt\right|\\
&\qquad=\left|\lambda_1\int_{\mathbb{R}_+}(z(t)-y(t)\overline{\widehat{x}_1(t)}\,dt+\lambda_2\int_{\mathbb{R}_+}z^{(n)}(t)\overline{\widehat{x}_1^{(n)}(t)}\,dt\right|\le\lambda_1\delta^2+\lambda_2.
\end{aligned}$$

Consequently,

$$E_k(W_2^n(\mathbb{R}_+),\delta)\le e_k(W_2^n(\mathbb{R}_+),\delta,\widehat{m})\le\lambda_1\delta^2+\lambda_2. \tag{10}$$

The last inequality together with (9) gives

$$\begin{aligned}
E_k(W_2^n(\mathbb{R}_+),\delta)&=\lambda_1\delta^2+\lambda_2\\
&=A_{nk}\left(\frac{2n}{2n-2k-1}\right)^{\frac{2n-2k-1}{4n}}\left(\frac{2n}{2k+1}\right)^{\frac{2k+1}{4n}}\delta^{\frac{2n-2k-1}{2n}}.
\end{aligned}$$

Inequality (10) implies also that $\widehat{m}$ is optimal method of recovery. $\square$

Note that the exact solution of extremal problem (8) gives us the exact inequality

$$|x^{(k)}(0)|\le K_{nk}\|x(\cdot)\|_{L_2(\mathbb{R}_+)}^{\frac{2n-2k-1}{2n}}\|x^{(n)}(\cdot)\|_{L_2(\mathbb{R}_+)}^{\frac{2k+1}{2n}},$$

where

$$K_{nk}=A_{nk}\left(\frac{2n}{2n-2k-1}\right)^{\frac{2n-2k-1}{4n}}\left(\frac{2n}{2k+1}\right)^{\frac{2k+1}{4n}}.$$

It may be also obtained from exact inequality (6) by Proposition 4 from [6, p. 119]

We now proceed to the Stechkin problem.

Theorem 2. *The following equality*

$$S_k(W_2^n(\mathbb{R}_+),N)=A_{nk}^{\frac{2n}{2k+1}}\sqrt{\frac{2k+1}{2n-2k-1}}\left(\frac{2n-2k-1}{2n}\right)^{\frac{n}{2k+1}}N^{-\frac{2n-2k-1}{2k+1}}$$

holds. The functional

$$\langle\widehat{y}^*,x(\cdot)\rangle=\beta_N^{k+1}\int_{\mathbb{R}_+}x(t)\overline{\widehat{x}(\beta_N t)}\,dt$$

where

$$\beta_N=\left(\frac{2n}{2n-2k-1}\right)^{\frac{1}{2k+1}}\left(\frac{N}{A_{nk}}\right)^{\frac{2}{2k+1}}$$

is extremal.

Proof. As was proved in the optimal recovery problem among all optimal methods there exists a method defined by a linear continuous functional, therefore

$$E_k(W_2^n(\mathbb{R}_+),\delta)=\inf_{N>0}\inf_{\|y^*\|\le N}\sup_{\substack{x(\cdot)\in W_2^n(\mathbb{R}_+),\ y(\cdot)\in L_2(\mathbb{R}_+)\\ \|x(\cdot)-y(\cdot)\|_{L_2(\mathbb{R}_+)}\le\delta}}|x^{(k)}(0)-\langle y^*,y(\cdot)\rangle|$$

$$\le\inf_{\|y^*\|\le N}\sup_{x(\cdot)\in W_2^n(\mathbb{R}_+)}|x^{(k)}(0)-\langle y^*,x(\cdot)\rangle|+\delta N=S_k(W_2^n(\mathbb{R}_+),N)+\delta N.$$

Consequently, for all $N>0$

$$S_k(W_2^n(\mathbb{R}_+),N)\ge E_k(W_2^n(\mathbb{R}_+),\delta)-\delta N. \tag{11}$$

We define the linear functional $\widehat{y}^*$ as follows

$$\langle\widehat{y}^*,x(\cdot)\rangle=\lambda_1\int_{\mathbb{R}_+}x(t)\overline{\widehat{x}_1(t)}\,dt.$$

Then $\|\widehat{y}^*\|=\lambda_1\delta$. If we choose δ such that $N=\lambda_1\delta$, then it follows from (11) that

$$S_k(W_2^n(\mathbb{R}_+),N)\ge\lambda_2.$$

On the other hand, in view of (7) we have

$$S_k(W_2^n(\mathbb{R}_+),N)\le\sup_{x(\cdot)\in W_2^n(\mathbb{R}_+)}|x^{(k)}(0)-\langle\widehat{y}^*,x(\cdot)\rangle|=\lambda_2.$$

Consequently, $S_k(W_2^n(\mathbb{R}_+),N)=\lambda_2$. If $N=\lambda_1\delta$, then

$$\delta=\delta_N=\sqrt{\frac{2n-2k-1}{2k+1}}\left(\frac{2n-2k-1}{2n}\right)^{\frac{n}{2k+1}}\left(\frac{A_{nk}}{N}\right)^{\frac{2n}{2k+1}}.$$

For $\delta=\delta_N$ we have

$$\lambda_2=A_{nk}^{\frac{2n}{2k+1}}\sqrt{\frac{2k+1}{2n-2k-1}}\left(\frac{2n-2k-1}{2n}\right)^{\frac{n}{2k+1}}N^{-\frac{2n-2k-1}{2k+1}}.$$

The functional $\widehat{y}^*$ may be written in the following way

$$\langle\widehat{y}^*,x(\cdot)\rangle=\beta^{k+1}\int_{\mathbb{R}_+}x(t)\overline{\widehat{x}(\beta t)}\,dt$$

where β is defined in Theorem 1. For $\delta=\delta_N$

$$\beta=\beta_N=\left(\frac{2n}{2n-2k-1}\right)^{\frac{1}{2k+1}}\left(\frac{N}{A_{nk}}\right)^{\frac{2}{2k+1}}.$$

□

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