EXTREMAL PROBLEMS OF THE HADAMAR THREE-CIRCLE THEOREM TYPES AND OPTIMAL RECOVERY OF OPERATORS

K. YU. OSIPENKO

The well-known Hadamard three-circle theorem states that if $f(z)$ is a holomorphic function on the annulus $r_1 \le |z| \le r_2$ and

$$M(r) = \max_{|z|=r} |f(z)|,$$

then

$$M(r) \le M(r_1)^{\frac{\log r_2/r}{\log r_2/r_1}} M(r_2)^{\frac{\log r/r_1}{\log r_2/r_1}}$$

for any three concentric circles of radii $r_1 < r < r_2$.

In 1913 E. Landau considered a very similar problem. He took derivatives instead of circles. He proved that for all functions $x(\cdot) \in L_\infty(\mathbb{R}_+)$ with the first derivative locally absolutely continuous on $\mathbb{R}_+$ and $x''(\cdot) \in L_\infty(\mathbb{R}_+)$ the following exact inequality

$$\|x'(\cdot)\|_{L_\infty(\mathbb{R}_+)} \le 2\|x(\cdot)\|^{1/2}_{L_\infty(\mathbb{R}_+)} \|x''(\cdot)\|^{1/2}_{L_\infty(\mathbb{R}_+)}$$

holds. Then in 1914 Hadamard solved the analogous problem for $\mathbb{R}$.

In 1934 Hardy, Littlewood, and Pólya proved that for all integers $0 < k < r$ the exact inequality

$$\|x^{(k)}(\cdot)\|_{L_2(\mathbb{R})} \le \|x(\cdot)\|^{1-\frac{k}{r}}_{L_2(\mathbb{R})} \|x^{(r)}(\cdot)\|^{\frac{k}{r}}_{L_2(\mathbb{R})} \tag{1}$$

holds for all functions $x(\cdot) \in L_2(\mathbb{R})$ for which the $(r-1)$-st derivative is locally absolute continuous on $\mathbb{R}$ and $x^{(r)}(\cdot) \in L_2(\mathbb{R})$.

The exact inequality (1) may be easily obtained passing to the Fourier transforms from the following extremal problem

$$\int_{\mathbb{R}} \xi^{2k} |Fx(\xi)|^2 \, d\xi \to \max, \quad \int_{\mathbb{R}} |Fx(\xi)|^2 \, d\xi \le \delta_1^2,$$
$$\int_{\mathbb{R}} \xi^{2n} |Fx(\xi)|^2 \, d\xi \le \delta_2^2,$$

This research was carried out with the financial support of the Russian Foundation for Basic Research (grant nos. 08-01-00450 and 08–01–90001).

where $Fx(\cdot)$ is the Fourier transform of $x(\cdot)$. The value of this extremal problem coincides with the value of the extremal problem

$$\int_{\mathbb{R}} \xi^{2k}|Fx(\xi)|^2\,d\xi \to \max, \quad \int_{|\xi|\le\sigma_1}|Fx(\xi)|^2\,d\xi \le \delta_1^2,$$
$$\int_{|\xi|\ge\sigma_2}\xi^{2n}|Fx(\xi)|^2\,d\xi \le \delta_2^2$$

for some $\sigma_1 \ge \sigma_2$. Using this fact we obtain a collection of optimal recovery methods of $x^{(k)}(\cdot)$ from inaccurate information about the Fourier transform of $x(\cdot)$.

References

[1] *Magaril-Il'yaev G. G., Osipenko K. Yu.* Optimal recovery of functions and their derivatives from inaccurate information about a spectrum and inequalities for derivatives, *Funkc. analiz i ego prilozh.*, 37 (2003), 51–64; English transl. in *Funct. Anal and Its Appl.*, 37 (2003), 203–214.

MATI — Russian State Technological University