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OPTIMAL RECOVERY OF SEMI-GROUP OPERATORS FROM INACCURATE DATA

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Abstract. The problem of optimal recovery of the operator at a given value of the parameter from inaccurate information about the other parameters is solved for a special one-parameter semi-group of operators. A family of optimal recovery methods is constructed. As a corollary, we obtain families of optimal recovery methods in the problem of recovery of a solution of the heat equation on the line and in the problem of recovery of a solution to the Dirichlet problem for the half-plane.

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1 Statement of problem and formulation of the main result

Let X be a real or complex Banach space with elements $f, g, \ldots$, and let $T(t)$, $0 \le t < \infty$, be a family of continuous linear operators on X to itself having the semi-group property: $T(t_1+t_2) = T(t_1)T(t_2)$, $t_1, t_2 \ge 0$.

We set the following problem: recover (in the best possible way) the value of operator $T(\tau)$ from approximate values of operators $T(t_1)$ and $T(t_2)$, where $\tau \ne t_i$, $i = 1, 2$.

A precise statement is as follows. Assume that for each $f \in X$ we know elements $g_i \in X$, $i = 1, 2$, such that

$$\|T(t_i)f - g_i\|_X \le \delta_i, \quad \delta_i > 0, \quad i = 1, 2.$$

Any map $\varphi\colon X \times X \to X$ we call a method of recovery. The error of this method is the value

$$e(\tau, \varphi) = \sup_{\substack{f\in X,\ g_i\in X,\ i=1,2\\ \|T(t_i)f-g_i\|_X\le\delta_i,\ i=1,2}} \|T(\tau)f - \varphi(g_1, g_2)\|_X.$$

We are interested in the value

$$E(\tau) = \inf_{\varphi} e(\tau, \varphi),$$

where the infimum is taken over all $\varphi\colon X \times X \to X$, is called the *optimal recovery error*, and methods $\widehat{\varphi}$, for which the lower bound is attained, i. e., for which

$$E(\tau) = e(\tau, \widehat{\varphi}),$$

will be called *optimal recovery methods.*

In this work for a certain family of operators, we compute the exact value of the optimal recovery error and find a family of optimal recovery methods.

Let $a(\cdot)$ be a continuous non-negative function on $\mathbb{R}$, $a(0)=0$ and $a(\xi)\to+\infty$ as $|\xi|\to+\infty$. Let F be the Fourier transform in $L_2(\mathbb{R})$. We define the family of operators $T_a(t)\colon L_2(\mathbb{R})\to L_2(\mathbb{R})$, $t\ge 0$, by the formula

$$F[T_a(t)f](\xi)=e^{-ta(\xi)}F[f](\xi)\quad\text{for a. e. }\xi\in\mathbb{R},\ \forall\, f(\cdot)\in L_2(\mathbb{R}).$$

These operators are obviously linear. We show that they are continuous. Indeed, let $\omega(\cdot)\in L_\infty(\mathbb{R})$ and the operator $\Lambda_\omega\colon L_2(\mathbb{R})\to L_2(\mathbb{R})$ such that $F[\Lambda_\omega f](\xi)=\omega(\xi)F[f](\xi)$ for a. e. $\xi\in\mathbb{R}$. Then, by Plancherel's theorem for any function $f(\cdot)\in L_2(\mathbb{R})$ we have

$$\begin{aligned}\|\Lambda_w f(\cdot)\|^2_{L_2(\mathbb{R})}&=\int_{\mathbb{R}}|\Lambda_w f(x)|^2\,dx=\frac{1}{2\pi}\int_{\mathbb{R}}|F[\Lambda_w f](\xi)|^2\,d\xi\\&=\frac{1}{2\pi}\int_{\mathbb{R}}|w(\xi)|^2|F[f](\xi)|^2\,d\xi\le\|w(\cdot)\|^2_{L_\infty(\mathbb{R})}\frac{1}{2\pi}\int_{\mathbb{R}}|F[f](\xi)|^2\,d\xi\\&=\|w(\cdot)\|^2_{L_\infty(\mathbb{R})}\int_{\mathbb{R}}|f(x)|^2\,dx=\|w(\cdot)\|^2_{L_\infty(\mathbb{R})}\|f(\cdot)\|^2_{L_2(\mathbb{R})},\end{aligned}$$

that is, the operator Λ_w is continuous.

Check the semi-group property. For any function $f(\cdot)\in L_2(\mathbb{R})$ we have by Plancherel's theorem for a. e. $\xi\in\mathbb{R}$.

$$\begin{aligned}F[T_a(t_1+t_2)f](\xi)&=e^{-(t_1+t_2)a(\xi)}F[f](\xi)=e^{-t_1a(\xi)}F[T_a(t_2)f](\xi)\\&=F[T_a(t_1)(T_a(t_2)f)](\xi).\end{aligned}$$

Since $F\colon L_2(\mathbb{R})\to L_2(\mathbb{R})$ is an isomorphism, it follows that $T_a(t_1+t_2)=T_a(t_1)T_a(t_2)$.

Let $0\le t_1<\tau<t_2$, $\delta_1>0$, $\delta_2>0$. Put

$$\widehat{\lambda}_1=\frac{t_2-\tau}{t_2-t_1}\left(\frac{\delta_1}{\delta_2}\right)^{\frac{-2(\tau-t_1)}{t_2-t_1}},\qquad \widehat{\lambda}_2=\frac{\tau-t_1}{t_2-t_1}\left(\frac{\delta_1}{\delta_2}\right)^{\frac{2(t_2-\tau)}{t_2-t_1}}.$$

The main result of this work is the following

Theorem 1.1. *Let $0\le t_1<\tau<t_2$ and $\delta_1>\delta_2>0$. Then*

$$E(\tau)=\sqrt{\widehat{\lambda}_1\delta_1^2+\widehat{\lambda}_2\delta_2^2}\,.$$

Function $\xi\mapsto\widehat{\lambda}_1e^{-2t_1a(\xi)}+\widehat{\lambda}_2e^{-2t_2a(\xi)}-e^{-2\tau a(\xi)}$ is non-negative for all $\xi\in\mathbb{R}$, and for any function $\omega(\cdot)\in L_\infty(\mathbb{R})$ such that for a. e. $\xi\in\mathbb{R}$

$$\begin{aligned}&\left|\omega(\xi)-\frac{\widehat{\lambda}_2e^{-(\tau-t_1)a(\xi)}}{\widehat{\lambda}_1e^{(t_2-t_1)a(\xi)}+\widehat{\lambda}_2e^{-(t_2-t_1)a(\xi)}}\right|\\&\qquad\le\frac{\sqrt{\widehat{\lambda}_1\widehat{\lambda}_2}\,e^{t_2a(\xi)}}{\widehat{\lambda}_1e^{(t_2-t_1)a(\xi)}+\widehat{\lambda}_2e^{-(t_2-t_1)a(\xi)}}\sqrt{\widehat{\lambda}_1e^{-2t_1a(\xi)}+\widehat{\lambda}_2e^{-2t_2a(\xi)}-e^{-2\tau a(\xi)}}\,,\end{aligned}$$

the method $\widehat{\varphi}_\omega\colon L_2(\mathbb{R})\times L_2(\mathbb{R})\to L_2(\mathbb{R})$ is defined in terms of Fourier transform for a. e. $\xi\in\mathbb{R}$ by the formula

$$F[\widehat{\varphi}_\omega(g_1,g_2)](\xi)=(e^{-(\tau-t_1)a(\xi)}-\omega(\xi)e^{-(t_2-t_1)a(\xi)})F[g_1](\xi)+\omega(\xi)F[g_2](\xi)$$

is optimal.

2 Proof of Theorem 1.1

The scheme of the proof is as follows. First, we prove an estimate from below for the optimal recovery error. Then we construct a family of recovery methods whose errors coincide with this estimate and thus these methods are optimal.

1) *The estimate from below for the optimal recovery error.* We prove this estimate for an abstract family of operators defined above.

Let us show that the value of the problem

$$\|T(\tau)f\|_X \to \max, \quad \|T(t_i)f\|_X \le \delta_i, \quad i = 1, 2, \quad f \in X, \tag{2.1}$$

i. e., the supremum of the functional to be maximized under the indicated constraints, is not greater than $E(\tau)$.

Indeed, let $f_0 \in X$ and $\|T(t_i)f_0\|_X \le \delta_i$, $i = 1, 2$. Then obviously $-f_0$ satisfies the same conditions and we have for an arbitrary method φ

$$\begin{aligned} 2\|T(\tau)f_0\|_X &= \|T(\tau)f_0 - \varphi(0,0) - (T(\tau)(-f_0) - \varphi(0,0))\|_X \\ &\le \|T(\tau)f_0 - \varphi(0,0)\|_X + \|T(\tau)(-f_0) - \varphi(0,0)\|_X \\ &\le 2 \sup_{\substack{f\in X,\\ \|T(t_i)f)\|_X\le\delta_i,\, i=1,2}} \|T(\tau)f - \varphi(0,0)\|_X \\ &\le 2 \sup_{\substack{f\in X,\ g_i\in X,\, i=1,2,\\ \|T(t_i)f-g_i\|_X\le\delta_i,\, i=1,2}} \|T(\tau)f - \varphi(g_1,g_2)\|_X = 2e(\tau,\varphi). \end{aligned}$$

Passing to the supremum over all $f \in X$ such that $\|T(t_i)f\|_X \le \delta_i$, $i = 1, 2$, on the left-hand side and to the infimum over all methods φ on the right-hand side, we obtain the inequality

$$\sup_{\substack{f\in X,\\ \|T(t_i)f\|_X\le\delta_i,\, i=1,2}} \|T(\tau)f\|_X \le E(\tau), \tag{2.2}$$

which means that the value of problem (2.1) is not greater than $E(\tau)$.

Now find the value of problem (2.1) for the family $T_a(t)$, $t \ge 0$. By Plancherel's theorem, we have

$$\int_{\mathbb{R}} |T_a(t)f(x)|^2\,dx = \frac{1}{2\pi}\int_{\mathbb{R}} |F[T_a(t)f](\xi)|^2\,d\xi = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-2ta(\xi)}|F[f](\xi)|^2\,d\xi.$$

Then the square of the value of problem (2.1) for the family $T_a(t)$, $t \ge 0$, is equal to the value of such problem

$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{-2\tau a(\xi)}|F[f](\xi)|^2\,d\xi \to \max, \ \frac{1}{2\pi}\int_{\mathbb{R}} e^{-2t_i a(\xi)}|F[f](\xi)|^2\,d\xi \le \delta_i^2, \quad i = 1, 2, \quad f(\cdot) \in L_2(\mathbb{R}). \tag{2.3}$$

This problem can be considered as a problem where the variables are positive measures $d\mu_f(\xi) = (2\pi)^{-1}|F[f](\xi)|^2\,d\xi$, $f(\cdot) \in L_2(\mathbb{R})$. It is convenient to consider a more general problem, namely, the extension of problem (2.3) to the set of all positive Borel measures μ on the line:

$$\int_{\mathbb{R}} e^{-2\tau a(\xi)}\,d\mu(\xi) \to \max, \quad \int_{\mathbb{R}} e^{-2t_1 a(\xi)}\,d\mu(\xi) \le \delta_1^2, \quad \int_{\mathbb{R}} e^{-2t_2 a(\xi)}\,d\mu(\xi) \le \delta_2^2, \quad \mu \ge 0. \tag{2.4}$$

This is a convex extremal problem. Its Lagrange function has the form

$$\mathcal{L}(\mu,\lambda) = -\lambda_0 \int_{\mathbb{R}} e^{-2\tau a(\xi)}\, d\mu(\xi) + \lambda_1 \left(\int_{\mathbb{R}} e^{-2t_1 a(\xi)} d\mu(\xi) - \delta_1^2\right) + \lambda_2 \left(\int_{\mathbb{R}} e^{-2t_2 a(\xi)} d\mu(\xi) - \delta_2^2\right),$$

where $\lambda = (\lambda_0, \lambda_1, \lambda_2)$ is the set of Lagrange multipliers.

If $\widehat{\mu}$ delivers a minimum in problem (2.4), then according to the Karush–Kuhn–Tucker theorem (see [1]) there is a set of Lagrange multipliers $\widehat{\lambda} = (\widehat{\lambda}_0, \widehat{\lambda}_1, \widehat{\lambda}_2)$, not all of which vanish, such that

(a) $\min\limits_{\mu\ge 0} \mathcal{L}(\mu, \widehat{\lambda}\,) = \mathcal{L}(\widehat{\mu}, \widehat{\lambda}\,)$;

(b) $\widehat{\lambda}_i \ge 0, \quad i = 0, 1, 2$;

(c) $\widehat{\lambda}_1 \left(\int_{\mathbb{R}} e^{-2t_1 a(\xi)}\, d\widehat{\mu}(\xi) - \delta_1^2\right) = 0,\ \ \widehat{\lambda}_2 \left(\int_{\mathbb{R}} e^{-2t_2\, (\xi)}\, d\widehat{\mu}(\xi) - \delta_2^2\right) = 0.$

We show that $\widehat{\lambda}_0 > 0$. Indeed, if $\widehat{\lambda}_0 = 0$, then $\widehat{\lambda}_1 + \widehat{\lambda}_2 > 0$ and we have for $\widehat{\lambda} = (0, \widehat{\lambda}_1, \widehat{\lambda}_2)$ and $\mu = 0$ taking into account condition (c)

$$\mathcal{L}(0, \widehat{\lambda}) = -\widehat{\lambda}_1 \delta_1^2 - \widehat{\lambda}_2 \delta_2^2 < 0 = \widehat{\lambda}_1 \left(\int_{\mathbb{R}} e^{-2t_1 a(\xi)} d\widehat{\mu}(\xi) - \delta_1^2\right) + \widehat{\lambda}_2 \left(\int_{\mathbb{R}} e^{-2t_2 a(\xi)} d\widehat{\mu}(\xi) - \delta_2^2\right) = \mathcal{L}(\widehat{\mu}, \widehat{\lambda})$$

that is impossible by virtue of condition (a).

If $\widehat{\lambda}_0 > 0$, then conditions (a), (b) and (c) are sufficient for $\widehat{\mu}$ to be a solution of problem (2.4). More precisely, if the measure $\widehat{\mu}$ is admissible in problem (2.4) and there is such a vector $\widehat{\lambda} = (\widehat{\lambda}_0, \widehat{\lambda}_1, \widehat{\lambda}_2)$ with $\widehat{\lambda}_0 > 0$ that conditions (a), (b) and (c) are satisfied, then $\widehat{\mu}$ is a solution of this problem.

In fact, let $\widehat{\mu}$ be an admissible measure in problem (2.4). Using this fact together with (b) and then (a) and (c) we obtain that

$$\begin{aligned} -\widehat{\lambda}_0 \int_{\mathbb{R}} e^{-2\tau a(\xi)}\, d\mu(\xi) &\ge -\widehat{\lambda}_0 \int_{\mathbb{R}} e^{-2\tau a(\xi)}\, d\mu(\xi) + \widehat{\lambda}_1 \left(\int_{\mathbb{R}} e^{-2t_1 a(\xi)} d\mu(\xi) - \delta_1^2\right) \\ &\quad + \widehat{\lambda}_2 \left(\int_{\mathbb{R}} e^{-2t_2 a(\xi)} d\mu(\xi) - \delta_2^2\right) = \mathcal{L}(\mu, \widehat{\lambda}) \ge \mathcal{L}(\widehat{\mu}, \widehat{\lambda}) - \widehat{\lambda}_0 \int_{\mathbb{R}} e^{-2\tau a(\xi)}\, d\widehat{\mu}(\xi) \\ &\quad + \widehat{\lambda}_1 \left(\int_{\mathbb{R}} e^{-2t_1 a(\xi)} d\widehat{\mu}(\xi) - \delta_1^2\right) + \widehat{\lambda}_2 \left(\int_{\mathbb{R}} e^{-2t_2 a(\xi)} d\widehat{\mu}(\xi) - \delta_2^2\right) \\ &= -\widehat{\lambda}_0 \int_{\mathbb{R}} e^{-2\tau a(\xi)}\, d\widehat{\mu}(\xi). \end{aligned}$$

Dividing by $\widehat{\lambda}_0$, we come to the fact that $\widehat{\mu}$ is a solution of problem (2.4).

We now investigate relations (a), (b) and (c) to understand the structure of measure $\widehat{\mu}$ and vector $\widehat{\lambda}$.

We rewrite the Lagrange function as

$$\mathcal{L}(\mu,\lambda)=\int_{\mathbb{R}}(-\lambda_0e^{-2\tau a(\xi)}+\lambda_1e^{-2t_1a(\xi)}+\lambda_2e^{-2t_2a(\xi)})\,d\mu(\xi)-\lambda_1\delta_1^2-\lambda_2\delta_2^2.$$

Then condition (a) is equivalent to the inequality

$$\int_{\mathbb{R}}(-\widehat{\lambda}_0e^{-2\tau a(\xi)}+\widehat{\lambda}_1e^{-2t_1a(\xi)}+\widehat{\lambda}_2e^{-2t_2a(\xi)})\,d\mu(\xi)$$
$$\geq\int_{\mathbb{R}}(-\widehat{\lambda}_0e^{-2\tau a(\xi)}+\widehat{\lambda}_1e^{-2t_1a(\xi)}+\widehat{\lambda}_2e^{-2t_2a(\xi)})\,d\widehat{\mu}(\xi)\quad(2.5)$$

for all $\mu\geq 0$.

Check that the value on the right must be zero. Indeed, if it is positive (negative), then taking $\mu=(1/2)\widehat{\mu}$ ($\mu=2\widehat{\mu}$), we come to a contradiction with (2.5).

Further, the expression under the integral sign in (2.5) must be non-negative. If at some point this expression takes a negative value, then taking as a measure μ Dirac measure at this point, we get a negative number in (2.5) on the left, which is impossible.

So, the vector $\widehat{\lambda}$ must be such that the function under the sign of the integral in (2.5) is nonnegative, and the measure $\widehat{\mu}$ must be concentrated in zeros of this function.

Let $\widehat{\mu}=A\delta_{\xi_0}$, where δ_{ξ_0} is the Dirac measure at the point $\xi_0\in\mathbb{R}$ and $A>0$. Choose ξ_0 and A so as to fulfill condition (c), and then choose positive $\widehat{\lambda}_0$, $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$ so that the function under the integral sign in (2.5) is nonnegative and equals zero at the point ξ_0.

So, take ξ_0 and A such that the following equalities hold:

$$\int_{\mathbb{R}}e^{-2t_ia(\xi)}\,d\widehat{\mu}(\xi)=Ae^{-2t_ia(\xi_0)}=\delta_i^2,\quad i=1,2.\qquad(2.6)$$

This implies that

$$a(\xi_0)=\frac{\ln(1/\delta_2)-\ln(1/\delta_1)}{t_2-t_1},\qquad A=\delta_1^{2t_2/(t_2-t_1)}\,\delta_2^{-2t_1/(t_2-t_1)}.\qquad(2.7)$$

Let us move on to the selection of Lagrange multipliers. To do this, we rewrite the expression under the sign of the integral in (2.5) as follows

$$e^{-2\tau a(\xi)}(-\widehat{\lambda}_0+\widehat{\lambda}_1e^{-2(t_1-\tau)a(\xi)}+\widehat{\lambda}_2e^{-2(t_2-\tau)a(\xi)}).\qquad(2.8)$$

It is clear that the nonnegativity of the function in brackets implies the nonnegativity of the whole expression. Define the function $\alpha\mapsto H(\alpha,\widehat{\lambda})$ on $\mathbb{R}$ by the formula

$$H(\alpha,\widehat{\lambda})=-\widehat{\lambda}_0+\widehat{\lambda}_1e^{-2(t_1-\tau)\alpha}+\widehat{\lambda}_2e^{-2(t_2-\tau)\alpha}$$

and assume that at the point $a(\xi_0)$ this function and its derivative are equal to zero. This gives the following relations

$$\begin{cases}\widehat{\lambda}_1e^{-2(t_1-\tau)a(\xi_0)}+\widehat{\lambda}_2e^{-2(t_2-\tau)a(\xi_0)}=\widehat{\lambda}_0,\\[2mm] \widehat{\lambda}_1(t_1-\tau)e^{-2(t_1-\tau)a(\xi_0)}+\widehat{\lambda}_2(t_2-\tau)e^{-2(t_2-\tau)a(\xi_0)}=0,\end{cases}$$

from which we obtain that

$$\widehat{\lambda}_1=\widehat{\lambda}_0\frac{t_2-\tau}{t_2-t_1}e^{2(t_1-\tau)a(\xi_0)},\quad\widehat{\lambda}_2=\widehat{\lambda}_0\frac{\tau-t_1}{t_2-t_1}e^{2(t_2-\tau)a(\xi_0)}.\qquad(2.9)$$

The above reasoning allows us to determine the measure admissible in problem (2.4) and the Lagrange multipliers satisfying conditions (a), (b) and (c).

Since $\delta_1 > \delta_2$ and $a(\xi) \to +\infty$ as $\xi \to \pm\infty$, then there is such $\xi_0 \in \mathbb{R}$ that the first relation in (2.7) is satisfied.

Then it is easy to check that with such $a(\xi_0)$ and with A as in (2.7), the relations (2.6) are true and hence the condition (c) is satisfied.

Let us put $\widehat{\lambda}_0 = 1$, and let $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$ be as in (2.9). It is obvious that $\widehat{\lambda}_i > 0$, $i = 1, 2$, and thus conditions (b) are fulfilled. If we substitute $a(\xi_0)$ from (2.7) into formulas for $\widehat{\lambda}_i > 0$, $i = 1, 2$, we obtain expressions for these Lagrange multipliers, which are given before the formulation of the theorem.

With such $\widehat{\lambda}_i$, $i = 0, 1, 2$, the function $\alpha \mapsto H(\alpha, \widehat{\lambda})$ is equal to zero together with its derivative at the point $a(\xi_0)$. But it is a convex function and therefore it is nonnegative everywhere. Thus, the expression in (2.8) is everywhere nonnegative and vanishes at the point ξ_0. Hence, the expression on the left in (2.5) is nonnegative for any $\mu \geq 0$ and vanishes for $\widehat{\mu}$. This is equivalent to the condition (a).

So, conditions (a), (b) and (c) are fulfilled and so $\widehat{\mu}$ is a solution of problem (2.4). Simple calculations taking into account (2.7) show that the value of this problem is as follows

$$\int_{\mathbb{R}} e^{-2\tau a(\xi)}\, d\widehat{\mu}(\xi) = Ae^{-2\tau a(\xi_0)} = \widehat{\lambda}_1\delta_1^2 + \widehat{\lambda}_2\delta_2^2.$$

It is clear that the value of problem (2.4) is not less than the value of problem (2.3). We show that in fact the values of these problems coincide. Indeed, consider a family of functions $\varphi_n(\cdot) \in L_2(\mathbb{R})$, $n \in \mathbb{N}$, such that

$$F[\varphi_n](\xi) = \begin{cases} \sqrt{2\pi n}\, \delta_1^{\frac{t_2}{t_2-t_1}}\, \delta_2^{-\frac{t_1}{t_2-t_1}}, & \xi \in [\xi_0, \xi_0 + 1/n], \\ 0, & \xi \notin [\xi_0, \xi_0 + 1/n]. \end{cases}$$

It is easy to check that the functions $\varphi_n(\cdot)$ are admissible in problem (2.3) and that

$$\frac{1}{2\pi} \int_{\mathbb{R}} e^{-2\tau a(\xi)} |F[\varphi_n](\xi)|^2\, d\xi \to \widehat{\lambda}_1\delta_1^2 + \widehat{\lambda}_2\delta_2^2$$

as $n \to \infty$. Thus, the values of problems (2.4) and (2.3) are the same.

Then according to inequality (2.2) we obtain the following estimate

$$E(\tau) \geq \sqrt{\widehat{\lambda}_1\delta_1^2 + \widehat{\lambda}_2\delta_2^2}\,. \tag{2.10}$$

2) *The upper bound of the optimal recovery error and optimal recovery methods.* Let us start by considering an abstract family of operators. We will look for optimal methods among continuous linear operators $\varphi\colon X \times X \to X$. It is easy to check that this is equivalent to

$$\varphi(g_1, g_2) = \Lambda_1 g_1 + \Lambda_2 g_2,$$

where $\Lambda_i\colon X \to X$, $i = 1, 2$, are continuous linear operators.

Let us estimate the error of this method, which is by definition equal to the value of the following problem:

$$\|T(\tau)f - \Lambda_1 g_1 - \Lambda_2 g_2\|_X \to \max, \quad \|T(t_i)f - g_i\|_X \leq \delta_i,$$
$$g_i \in X, \quad i = 1, 2, \quad f \in X. \tag{2.11}$$

Let $f \in X$ and $g_i = T(t_i)f$, $i = 1, 2$. Then the triple (f, g_1, g_2) is admissible in this problem. For this triple, using the semigroup property of operators, the value of the maximized functional in the problem (2.11) takes the form

$$\begin{aligned}\|T(\tau)f - \Lambda_1 g_1 - \Lambda_2 g_2\|_X &= \|T(\tau)f - \Lambda_1 T(t_1)f - \Lambda_2 T(t_1)f\|_X \\ &= \|T(\tau - t_1)T(t_1)f - \Lambda_1 T(t_1)f - \Lambda_2 T(t_2 - t_1)T(t_1)f\|_X \\ &= \|(T(\tau - t_1) - \Lambda_1 - \Lambda_2 T(t_2 - t_1))T(t_1)f\|_X.\end{aligned}$$

If the operator in parentheses is not zero, the expression can be arbitrarily large. Therefore, the error of the method with such operators is equal $+\infty$. Since we are interested in optimal recovery methods, methods with infinite error can be discarded.

So, operators Λ_i, $i = 1, 2$, should be such that

$$T(\tau - t_1) - \Lambda_1 - \Lambda_2 T(t_2 - t_1) = 0.$$

If this condition holds, then denoting $\Lambda = \Lambda_2$ the maximized functional in problem (2.11) is written as

$$\begin{aligned}&\|T(\tau)f - (T(\tau - t_1) - \Lambda T(t_2 - t_1))g_1 - \Lambda g_2\|_X \\ &\qquad = \|(T(\tau - t_1) - \Lambda T(t_2 - t_1))(T(t_1)f - g_1) + \Lambda(T(t_2)f - g_2)\|_X.\end{aligned}$$

Thus, we come to the following problem, the value of which we are interested in

$$\begin{aligned}&\|(T(\tau - t_1) - \Lambda T(t_2 - t_1))(T(t_1)f - g_1) + \Lambda(T(t_2)f - g_2)\|_X \to \max, \\ &\qquad \|T(t_i)f - g_i\|_X \le \delta_i, \quad g_i \in X, \quad i = 1, 2, \quad f \in X. \quad (2.12)\end{aligned}$$

Let us now move on to the family of operators $T_a(t)$, $t \ge 0$. Because we recover an operator whose action in terms of Fourier transforms is multiplication by a function belonging to $L_\infty(\mathbb{R})$, then it is natural to search for optimal methods among such operators.

So, for each function $\omega(\cdot) \in L_\infty(\mathbb{R})$ we consider the operator $\Lambda_\omega \colon L_2(\mathbb{R}) \to L_2(\mathbb{R})$ whose Fourier transforms are defined as

$$F[\Lambda_\omega g](\xi) = \omega(\xi)F[g](\xi), \text{ for a. e. } \xi \in \mathbb{R}, \ \forall\, g \in L_2(\mathbb{R}).$$

It has been shown above that such an operator is continuous.

It is easy to see that in terms of Fourier transforms the squared value of problem (2.12) is equal to the value of the following problem

$$\begin{aligned}&\frac{1}{2\pi}\int_{\mathbb{R}} |(e^{-(\tau - t_1)a(\xi)} - \omega(\xi)e^{-(t_2 - t_1)a(\xi)})F[T_a(t_1)f - g_1](\xi) \\ &\qquad + \omega(\xi)F[T_a(t_2)f - g_2](\xi)|^2\, d\xi \to \max, \\ &\frac{1}{2\pi}\int_{\mathbb{R}} |F[T_a(t_i)f - g_i](\xi)|^2\, d\xi \le \delta_i^2, \quad g_i(\cdot) \in L_2(\mathbb{R}), \quad i = 1, 2, \\ &\qquad f(\cdot) \in L_2(\mathbb{R}). \quad (2.13)\end{aligned}$$

Let us estimate the maximized functional from above denoting for short

$$\omega_1(\xi) = e^{-(\tau - t_1)a(\xi)} - \omega(\xi)e^{-(t_2 - t_1)a(\xi)}.$$

Let $\widehat{\lambda}_i$, $i=1,2$ be Lagrange multipliers defined above. Then we have, using the CauchyпїSSchwarz inequality, for a. e. $\xi\in\mathbb{R}$

$$|\omega_1(\xi)F[T_a(t_1)f-g_1](\xi)+\omega(\xi)F[T_a(t_2)f-g_2](\xi)|^2$$
$$=\left|\frac{\omega_1(\xi)}{\sqrt{\widehat{\lambda}_1}}\sqrt{\widehat{\lambda}_1}\,F[T_a(t_1)f-g_1](\xi)+\frac{\omega(\xi)}{\sqrt{\widehat{\lambda}_2}}\sqrt{\widehat{\lambda}_2}\,F[T_a(t_2)f-g_2](\xi)\right|^2$$
$$\le\left(\frac{|\omega_1(\xi)|^2}{\widehat{\lambda}_1}+\frac{|\omega(\xi)|^2}{\widehat{\lambda}_2}\right)(\widehat{\lambda}_1|F[T_a(t_1)f-g_1](\xi)|^2+\widehat{\lambda}_2|F[T_a(t_2)f-g_2](\xi)|^2).$$

Let $S(\cdot)$ denote the function in large parentheses. It is clear that $S(\cdot)\in L_\infty(\mathbb{R})$. Integrating the last inequality and taking into account the constraints in problem (2.13), we find that the value of this problem does not exceed

$$\|S(\cdot)\|_{L_\infty(\mathbb{R})}(\widehat{\lambda}_1\delta_1^2+\widehat{\lambda}_2\delta_2^2).$$

Comparing this inequality with inequality (2.10) we see that if there is $\omega(\cdot)\in L_\infty(\mathbb{R})$ such that $\|S(\cdot)\|_{L_\infty(\mathbb{R})}\le 1$, then there is a linear optimal method $\widehat{\varphi}$, which in Fourier transforms acts by formula

$$F[\widehat{\varphi}(g_1,g_2)](\xi)=(e^{-(\tau-t_1)a(\xi)}-\omega(\xi)e^{-(t_2-t_1)a(\xi)})F[g_1](\xi)+\omega(\xi)F[g_2](\xi)$$

for a. e. $\xi\in\mathbb{R}$.

We will show that such functions $\omega(\cdot)\in L_\infty(\mathbb{R})$ exist. Condition $\|S(\cdot)\|_{L_\infty(\mathbb{R})}\le 1$ is equivalent to the inequality

$$\frac{|e^{-(\tau-t_1)a(\xi)}-\omega(\xi)e^{-(t_2-t_1)a(\xi)}|^2}{\widehat{\lambda}_1}+\frac{|\omega(\xi)|^2}{\widehat{\lambda}_2}\le 1$$

for a. e. $\xi\in\mathbb{R}$. This, in turn, (after squaring) is equivalent to the inequality

$$\left|\omega(\xi)-\frac{\widehat{\lambda}_2e^{-(\tau-t_1)a(\xi)}}{\widehat{\lambda}_1e^{(t_2-t_1)a(\xi)}+\widehat{\lambda}_2e^{-(t_2-t_1)a(\xi)}}\right|$$
$$\le\frac{\sqrt{\widehat{\lambda}_1\widehat{\lambda}_2}\,e^{t_2a(\xi)}}{\widehat{\lambda}_1e^{(t_2-t_1)a(\xi)}+\widehat{\lambda}_2e^{-(t_2-t_1)a(\xi)}}\sqrt{\widehat{\lambda}_1e^{-2t_1a(\xi)}+\widehat{\lambda}_2e^{-2t_2a(\xi)}-e^{-2\tau a(\xi)}}\,.$$

The expression on the right makes sense because the function under the root sign is the nonnegative function (2.8). Thus, the functions $\omega(\cdot)\in L_\infty(\mathbb{R})$ satisfying the condition $\|S(\cdot)\|_{L_\infty(\mathbb{R})}\le 1$ exist and thus the theorem is proved.

3 Examples

3.1 Optimal recovery of a solution of the heat equation

Consider the problem of optimal recovery of temperature on $\mathbb{R}$ at time instant τ from its approximate measurements at time instants t_1 and t_2. The heat propagation on line is described by the equation

$$\begin{cases}\dfrac{\partial u}{\partial t}=\dfrac{\partial^2 u}{\partial x^2}\,,\\ u(0,\cdot)=f(\cdot).\end{cases}$$

We assume that $f(\cdot)\in L_2(\mathbb{R})$. The unique solution of this problem for $t>0$ is given by the Poisson integral

$$u(t,x)=u(t,x;f)=\frac{1}{2\sqrt{\pi t}}\int_{\mathbb{R}}e^{-\frac{|x-y|^2}{4t}}f(y)\,dy$$

with $u(t,\cdot) \to f(\cdot)$ as $t \to 0$ in the metric of $L_2(\mathbb{R})$.

The Fourier transform of the solution of the heat equation is given by

$$F[u(t,x;f)](\xi) = e^{-t\xi^2} F[f](\xi)$$

(see, for example, [2]). Thus, our problem is a particular case of the problem considered above when $\alpha(\xi) = \xi^2$.

3.2 Optimal recovery of a solution of the Dirichlet problem

Consider the Dirichlet problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + \dfrac{\partial^2 u}{\partial x^2} = 0, \\ u(0,\cdot) = f(\cdot) \end{cases}$$

of finding a harmonic function $u(\cdot,\cdot)$ in the upper half-plane $(0,+\infty) \times \mathbb{R}$ such that $u(t,\cdot) \in L_2(\mathbb{R})$ for all $t > 0$, $u(t,\cdot) \to f(\cdot)$ as $t \to 0$ in the metric $L_2(\mathbb{R})$ and

$$\sup_{t>0} \|u(t,\cdot)\|_{L_2(\mathbb{R})} < \infty.$$

In this case, the solution of the problem is unique and is given by the Poisson integral

$$u(t,x) = u(t,x;f) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{tf(y)}{(x-y)^2 + t^2}\, dy.$$

(see [3]). The Fourier transform of the solution of the Dirichlet problem is given by

$$F[u(t,x;f)](\xi) = e^{-t|\xi|} F[f](\xi).$$

If we set the problem of optimal recovery of the solution of the Dirichlet problem on the line $t = \tau$ from inaccurate measurements of this solution on the lines $t = t_1$ and $t = t_2$, $t_1 < \tau < t_2$, we obtain a special case of the problem considered above.

Let us make a few concluding remarks. The first results related to the problem of optimal recovery of linear operators were obtained in [4]. This direction was further developed in the papers [5-7], where authors applied an approach based on the general principles of extremum theory. The application of the theory of optimal reconstruction of linear operators to problems of mathematical physics is described in [8-11].

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