On the Hardy-Littlewood-Pólya Inequality for Analytic Functions

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Hardy, Littlewood, and Pólya proved in 1934 that for all integer $0 < k < r$ and all functions $x(\cdot) \in L_2(\mathbb{R})$ such that the $(r-1)$-st derivative is locally absolutely continuous on $\mathbb{R}$ and $x^{(r)}(\cdot) \in L_2(\mathbb{R})$ the following exact inequality

$$\|x^{(k)}(\cdot)\|_{L_2(\mathbb{R})} \le \|x(\cdot)\|_{L_2(\mathbb{R})}^{1-\frac{k}{r}} \|x^{(r)}(\cdot)\|_{L_2(\mathbb{R})}^{\frac{k}{r}}$$

holds. We consider the analogue of this inequality for functions which are analytic in the strip $S_\beta = \{z \in \mathbb{C} : |\operatorname{Im} z| < \beta\}$.

Denote by $\mathcal{H}_2^\beta$ the Hardy space of functions $f(\cdot)$ analytic in the strip S_β and satisfying the condition

$$\|f(\cdot)\|_{\mathcal{H}_2^\beta} = \left(\sup_{0 \le \eta < \beta} \frac{1}{2} \int_{\mathbb{R}} (|f(t+i\eta)|^2 + |f(t-i\eta)|^2)\, dt \right)^{1/2} < \infty.$$

The Hardy-Sobolev space $\mathcal{H}_2^{r,\beta}$ is the space of functions $f(\cdot)$ analytic in the strip S_β for which $f^{(r)}(\cdot) \in \mathcal{H}_2^\beta$. Denote by $\mu_{r\beta}(x)$ the unique solution of the equation $t^r\sqrt{\operatorname{ch} 2\beta t} = x$ from the interval $[0, +\infty)$.

Theorem. *For all $r, k \in \mathbb{N}$, $k \le r$, and $\gamma_1, \gamma_2 > 0$*

$$\sup_{\substack{f(\cdot) \in \mathcal{H}_2^{r,\beta} \cap L_2(\mathbb{R}) \\ \|f^{(r)}(\cdot)\|_{\mathcal{H}_2^\beta} \le \gamma_1 \\ \|f(\cdot)\|_{L_2(\mathbb{R})} \le \gamma_2}} \|f^{(k)}(\cdot)\|_{L_2(\mathbb{R})} = \gamma_2 \mu_{r\beta}^k \left(\frac{\gamma_1}{\gamma_2} \right).$$

In other words, for all functions $f(\cdot) \in \mathcal{H}_2^{r,\beta} \cap L_2(\mathbb{R})$ which are not equivalent to zero the following exact inequality

$$\|f^{(k)}(\cdot)\|_{L_2(\mathbb{R})} \le \|f(\cdot)\|_{L_2(\mathbb{R})} \mu_{r\beta}^k \left(\frac{\|f^{(r)}(\cdot)\|_{\mathcal{H}_2^\beta}}{\|f(\cdot)\|_{L_2(\mathbb{R})}} \right)$$

holds.

The research was carried out with the financial support of the Russian Foundation for Basic Research (grant nos. 05-01-00275 and 05-01-00261), the President Grant for State Support of Leading Scientific Schools in Russian Federation (grant no. NSH-304.2003.1), and the Program "Universities of Russia" (grant nos. UR.03.01.130 and UR.04.02.536))