Optimal recovery of linear operators

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Let X be a linear space, $Y_1, \dots, Y_n$ be linear spaces with semi-inner products, $I_j\colon X \to Y_j$, $j = 1, \dots, n$ be linear operators, and Z be a normed linear space. We consider the problem of optimal recovery of a linear operator $T\colon X \to Z$ on a set

$$W = \{\, x \in X : \|I_j x\|_{Y_j} \le \delta_j,\ j = 1, \dots, k,\ k < n \,\}$$

by inaccurate information about values of the operators $I_{k+1}, \dots, I_n$. More precisely, we are interesting in the value

$$E(T, W, I, \delta) = \inf_{\varphi\colon Y_{k+1}\times\ldots\times Y_n \to Z} \ \sup_{\substack{x\in W,\ (y_{k+1},\dots,y_n)\in Y_{k+1}\times\ldots\times Y_n \\ \|I_j x - y_j\|_{Y_j}\le\delta_j,\ j=k+1,\dots,n}} \|Tx - \varphi(y)\|_Z,$$

and in a method $\widehat{\varphi}$ for which this infimum is attained (we call it an optimal method of recovery).

Consider the following extremal problem

$$(1) \qquad \|Tx\|_Z^2 \to \max, \quad \|I_j x\|_{Y_j} \le \delta_j^2,\ j = 1, \dots, n,\ x \in X.$$

Denote by

$$\mathcal{L}(x, \lambda) = -\|Tx\|_Z^2 + \sum_{j=1}^n \lambda_j \|I_j x\|_{Y_j}^2$$

the Lagrange function of this problem.

Theorem 1. *Suppose that there exist nonnegative* $\widehat{\lambda}_j$, $j = 1, \dots, n$ *such that* $\mathcal{L}(x, \widehat{\lambda}) \ge 0$ *for all* $x \in X$. *Let* $\{x_m\}$ *be a sequence of admissible elements in* (1) *such that*

$$(a) \quad \lim_{m\to\infty} \mathcal{L}(x_m, \widehat{\lambda}) = 0,$$

$$(b) \quad \lim_{m\to\infty} \sum_{j=1}^n \widehat{\lambda}_j (\|I_j x_m\|_{Y_j}^2 - \delta_j^2) = 0.$$

If for all $y = (y_{k+1}, \dots, y_n) \in Y_{k+1} \times \ldots \times Y_n$ *there exists an element* x_y *which is a solution of the problem*

$$\sum_{j=1}^k \widehat{\lambda}_j \|I_j x\|_{Y_j}^2 + \sum_{j=k+1}^n \widehat{\lambda}_j \|I_j x - y_j\|_{Y_j}^2 \to \min, \quad x \in X,$$

then the method

$$\widehat{\varphi}(y) = Tx_y$$

is optimal and

$$E(T, W, I, \delta) = \sqrt{\sum_{j=1}^n \widehat{\lambda}_j \delta_j^2}.$$

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