On a problem of optimal recovery and Kolmogorov type inequalities on an interval

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Abstract

We consider an optimal recovery problem for the k-th derivative of the function on an interval from the information on the function itself, given in the mean square metric. As a consequence of the solution we prove one Kolmogorov type inequality for derivatives on an interval and demonstrate that the constant in this inequality can be reduced by considering particular subsets of the function class.

Keywords: optimal recovery, inequality for derivatives, exact constant

Optimal recovery problem first appeared in the paper of Smolyak [1] and has been widely developed in [2]–[5]. The problems of this kind are also considered in [6]. Based on the general principles of extremal problems the new approach can be found in [7] - [10], as well as some results in this area. In the papers [11], [12] authors obtained some inequalities for derivatives and showed, that the problem of finding the exact constants in such inequalities can be formulated and efficiently solved as the corresponding optimal recovery problem. In this paper we develop their approach and prove one Kolmogorov type inequality for derivatives (originally obtained in [13] and discussed in the paragraph 5.3 of the book [14]) as a consequence of the solution of the optimal recovery problem. Moreover, we show that the constant in this inequality, which is accurate on the whole class of functions, may be reduced on its subsets. We give explicit expressions for these subsets and the corresponding constants.

Consider the space $L_2(\omega_\alpha, [-1,1])$ of measurable functions on $[-1,1]$, satisfying condition

$$\|x\|_{L_2(\omega_\alpha,[-1,1])} = \left(\int_{-1}^{1} w_\alpha(t)|x(t)|^2 dt\right)^{1/2} < \infty, \quad \omega_\alpha(t) = (1-t^2)^\alpha.$$

Denote by W^r the weighted Sobolev class, consisting of functions $x \in L_2([-1,1])$ with absolutely continuous $(r-1)$-derivative on $[-1,1]$ and

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$\|x^{(r)}\|_{L_2(w_r,[-1,1])} \le 1$, $r \in N$. Suppose that for a function $x \in W^r$ we know an approximation $g \in L_2([-1,1])$, such that $\|x-g\|_{L_2([-1,1])} \le \delta$, $\delta > 0$. On this information we want to recover the k-th derivative of x as an element of $L_2(\omega_k,[-1,1])$, where $0 \le k < r$. An arbitrary map $m : L_2([-1,1]) \to L_2(w_k,[-1,1])$ is called a method m of recovery of $x^{(k)}$. Define the error $e(\delta,m)$ of the method by

$$e(\delta,m) = \sup_{\substack{x\in W^r,\quad g\in L_2([-1,1]) \\ \|x-g\|_{L_2([-1,1])}\le\delta}} \|x^{(k)} - m(g)\|_{L_2(w_k,[-1,1])}.$$

Next, define the error $E(\delta)$ of the optimal recovery by

$$E(\delta) = \inf_{m:L_2([-1,1])\to L_2(w_k,[-1,1])} e(\delta,m). \tag{1}$$

The method of recovery m is optimal if the error of the optimal recovery $E(\delta)$ is achieved by the error $e(\delta,m)$ of m, i.e. $e(\delta,m)=E(\delta)$.

Consider Jacobi polynomials $\{P_l^\alpha\}_{l=0}^\infty$, $\alpha > -1$, which are orthogonal on $[-1,1]$ with respect to the weight $(1-t^2)^\alpha$. It's known ([15]), that

$$\int_{-1}^{1}(1-t^2)^\alpha P_l^\alpha(t)P_k^\alpha(t)dt = \begin{cases} 0, & k \ne l \\ \frac{2^{2\alpha+1}}{2l+2\alpha+1}\frac{(l+\alpha)!^2}{(l+2\alpha)!l!}, & k = l. \end{cases}$$

We set $Y_l^\alpha(t) = \sqrt{\frac{2l+2\alpha+1}{2^{2\alpha+1}}\frac{(l+2\alpha)!l!}{(l+\alpha)!^2}}P_l^\alpha(t)$ and construct an orthonormal basis $\{Y_l^\alpha\}_{l=0}^\infty$ in $L_2(w_\alpha,[-1,1])$, $\alpha > -1$.

Consider the set of points $\{(x_l,y_l)\}_{l=k}^\infty$, given by the formulas

$$x_l = \begin{cases} 0, & k \le l < r, \\ \frac{(l+r)!}{(l-r)!}, & l \ge r, \end{cases} \qquad y_l = \frac{(l+k)!}{(l-k)!}.$$

Let $x_s < \delta^{-2} \le x_{s+1}$, $s \ge r-1$ and put

$$\widehat{\lambda}_1 = \frac{y_{s+1}-y_s}{x_{s+1}-x_s}, \quad \widehat{\lambda}_2 = \frac{y_s x_{s+1} - y_{s+1}x_s}{x_{s+1}-x_s}. \tag{2}$$

Later we'll see , that $\widehat{\lambda}_1 \ge 0$ and $\widehat{\lambda}_2 > 0$.

Theorem 1. *The error of the optimal recovery is given by*

$$E(\delta) = \sqrt{\widehat{\lambda}_1 + \widehat{\lambda}_2\delta^2}$$

and the following methods are optimal

$$m_a(g)(t) = \sum_{l=k}^{r-1} g_l\sqrt{\frac{(l+k)!}{(l-k)!}}Y_{l-k}^k(t) + \sum_{l=r}^{\infty} a_l g_l\sqrt{\frac{(l+k)!}{(l-k)!}}Y_{l-k}^k(t), \tag{3}$$

where

$$g_l = \int_{-1}^{1} g(t) Y_l^0(t)dt, \tag{4}$$

$$a_l = \frac{\widehat{\lambda}_2}{\widehat{\lambda}_1 x_l + \widehat{\lambda}_2} + \epsilon_l \frac{\sqrt{\widehat{\lambda}_1 \widehat{\lambda}_2}}{\widehat{\lambda}_1 x_l + \widehat{\lambda}_2} \sqrt{\frac{x_l}{y_l}} \sqrt{x_l \widehat{\lambda}_1 + \widehat{\lambda}_2 - y_l}, \tag{5}$$

ϵ_l — *arbitrary numbers in* $[-1; 1]$.

Proof. Consider the extremal problem

$$\begin{gathered} \|x^{(k)}\|^2_{L_2(w_k,[-1,1])} \to \max, \\ \|x^{(r)}\|^2_{L_2(w_r,[-1,1]))} \leq 1, \quad \|x\|^2_{L_2([-1,1])} \leq \delta^2, \end{gathered} \tag{6}$$

which is called the dual problem to (1). Its solution gives the lower bound for $E(\delta)$. Indeed, for an arbitrary method m

$$\begin{aligned} e(\delta, m) = & \sup_{\substack{x \in W^r, \quad g \in L_2([-1,1]) \\ \|x-g\|_{L_2([-1,1])} \leq \delta}} \|m(g) - x^{(k)}\|_{L_2(w_k,[-1,1])} \geq \\ & \geq \sup_{\substack{x \in W^r \\ \|x\|_{L_2([-1,1])} \leq \delta}} \|m(0) - x^{(k)}\|_{L_2(w_k,[-1,1])} \geq \\ \geq & \sup_{\substack{x \in W^r \\ \|x\|_{L_2([-1,1])} \leq \delta}} \frac{\|m(0) - x^{(k)}\|_{L_2(w_k,[-1,1])} + \| - m(0) - x^{(k)}\|_{L_2(w_k,[-1,1])}}{2} \geq \\ & \geq \sup_{\substack{x \in W^r \\ \|x\|_{L_2([-1,1])} \leq \delta}} \|x^{(k)}\|_{L_2(w_k,[-1,1])}. \end{aligned}$$

The inequalities are true due to the central symmetry of the set of admissible functions. Hence

$$E(\delta) \geq \sup_{\substack{x \in W^r \\ \|x\|_{L_2([-1,1])} \leq \delta}} \|x^{(k)}\|_{L_2(w_k,[-1,1])}.$$

Consider the decomposition of x in the basis $\{Y_l^0\}_{l=0}^{\infty}$, which has the form $x(t) = \sum_{l=0}^{\infty} c_l Y_l^0(t)$. We use the formula (that follows from the formula (4.5.5) from [16])

$$\frac{d^k}{dt^k} P_l^{\alpha}(t) = \frac{(2\alpha + l + k)!}{2^k (2\alpha + l)!} P_{l-k}^{\alpha+k}(t)$$

to obtain $x^{(k)}(t) = \sum_{l=k}^{\infty} c_l \sqrt{\frac{(l+k)!}{(l-k)!}} Y_{l-k}^k(t)$, which is the decomposition of $x^{(k)}$ in the corresponding basis in $L_2(w_k, [-1, 1])$. The same decomposition takes place for $x^{(r)}$. Denote $c_l^2 = u_l$ and use Parseval identity to write the problem (6) in the following form

$$\sum_{l=k}^{\infty} u_l \tfrac{(l+k)!}{(l-k)!} \to \max,$$
$$\sum_{l=r}^{\infty} u_l \tfrac{(l+r)!}{(l-r)!} \le 1, \quad \sum_{l=0}^{\infty} u_l \le \delta^2, \quad u_l \ge 0, l = 0, \dots \ . \tag{7}$$

We write its Lagrange function, putting $u_l = 0$, $l = 0, \dots, k-1$, as these coefficients aren't included in the functional and thereby the second constraint in (7) may be equivalently presented as $\sum_{l=k}^{\infty} u_l \le \delta^2$.

$$L(u, \lambda_1, \lambda_2) = -\lambda_1 - \lambda_2\delta^2 + \sum_{l=k}^{\infty} u_l \left(-\frac{(l+k)!}{(l-k)!} + \lambda_2 \right) + \sum_{l=r}^{\infty} \lambda_1 u_l \frac{(l+r)!}{(l-r)!} =$$
$$= -\lambda_1 - \lambda_2\delta^2 + \sum_{l=k}^{\infty} u_l \left(\lambda_1 x_l + \lambda_2 - y_l \right), \quad u = (0, \dots, u_k, u_{k+1}, \dots).$$

If there exist Lagrange multipliers $\widehat{\lambda}_1$,$\widehat{\lambda}_2 \ge 0$ and element $\widehat{u}$, admissible in (7), that minimizes Lagrange function

$$\min_{u \ge 0} L(u, \widehat{\lambda}_1, \widehat{\lambda}_2) = L(\widehat{u}, \widehat{\lambda}_1, \widehat{\lambda}_2)$$

and satisfies

$$\widehat{\lambda}_1 \left(\sum_{l=r}^{\infty} \widehat{u}_l x_l - 1 \right) + \widehat{\lambda}_2 \left(\sum_{l=0}^{\infty} \widehat{u}_l - \delta^2 \right) = 0$$

(complementary slackness condition), then $\widehat{u}$ brings maximum to (7). This follows from the fact that from non–negativity of Lagrange multipliers, for all admissible u we have the inequality

$$L(u, \widehat{\lambda}_1, \widehat{\lambda}_2) \le -\sum_{l=k}^{\infty} u_l \frac{(l+k)!}{(l-k)!},$$

which implies

$$\min_{u \ge 0} L(u, \widehat{\lambda}_1, \widehat{\lambda}_2) \le \min_{\substack{u \ge 0 \\ \sum_{l=0}^{\infty} u_l \le \delta^2 \\ \sum_{l=r}^{\infty} u_l x_l \le 1}} -\sum_{l=k}^{\infty} u_l \frac{(l+k)!}{(l-k)!}.$$

From the fact, that $\widehat{u}$ minimizes Lagrange function and satisfies the complementary slackness condition it follows, that

$$\min_{u \ge 0} L(u, \widehat{\lambda}_1, \widehat{\lambda}_2) = -\sum_{l=k}^{\infty} \widehat{u}_l \frac{(l+k)!}{(l-k)!}.$$

Hence,

$$-\sum_{l=k}^{\infty} \widehat{u}_l \frac{(l+k)!}{(l-k)!} \le \min_{\substack{u_l \ge 0 \\ \sum_{l=0}^{\infty} u_l \le \delta^2 \\ \sum_{l=r}^{\infty} u_l x_l \le 1}} -\sum_{l=k}^{\infty} u_l \frac{(l+k)!}{(l-k)!},$$

i.e. $\widehat{u}$ is the solution to (7). We shall present such $\widehat{\lambda}_1$,$\widehat{\lambda}_2$ and $\widehat{u}$.

Consider expression $(y_{l+1}-y_l)/(x_{l+1}-x_l)$, which is the slope of the line, connecting the adjacent points of the set $\{(x_l,y_l)\}_{l=r-1}^{\infty}$. It decreases with the growth of l. For $l \geq r+1$ it's easily verified that

$$\frac{y_{l+1}-y_l}{x_{l+1}-x_l} = \frac{y_l-y_{l-1}}{x_l-x_{l-1}}\frac{l-r+1}{l-k+1}\frac{l+k}{l+r} \leq \frac{y_l-y_{l-1}}{x_l-x_{l-1}}$$

and the same holds true for $l=r$ as well.

The fact that the slope decreases as the sequences x_l and y_l increase monotonically to infinity implies that any line, connecting the adjacent points of the set $\{(x_l,y_l)\}_{l=r-1}^{\infty}$, is a support line to the given set, and the whole set of points $\{(x_l,y_l)\}_{l=k}^{\infty}$ lies entirely below such line (for $\{(x_l,y_l)\}_{l=k}^{r-2}$ the proposition is obvious). Also, there exists $s \geq r-1$, such that $x_s < \delta^{-2} \leq x_{s+1}$. Taking the line $y=\widehat{\lambda}_1 x+\widehat{\lambda}_2$ (where $\widehat{\lambda}_1$, $\widehat{\lambda}_2$ are defined in (2)), which connects the points (x_s,y_s) and (x_{s+1},y_{s+1}), we come to $\widehat{\lambda}_1 x_l+\widehat{\lambda}_2-y_l \geq 0, \quad l \geq k$. Hence

$$L(u,\widehat{\lambda}_1,\widehat{\lambda}_2) \geq -\widehat{\lambda}_1-\widehat{\lambda}_2\delta^2, \quad \forall u \geq 0.$$

Obviously, $\widehat{\lambda}_1 \geq 0$ as a slope of the line ($\widehat{\lambda}_1=0$ in case $k=0$). Also $\widehat{\lambda}_2>0$ being a value of the line at 0, which is bigger, than $y_l>0$ for $l=k,\dots,r-1$, as the set $\{(x_l,y_l)\}_{l=k}^{r-1}$ (where $x_l=0$) lies below the line. Consider the element $\widehat{u}$,

$$\widehat{u}_i = \begin{cases} 0, & i \notin \{s,s+1\}, \\ (\delta^2 x_{s+1}-1)/(x_{s+1}-x_s), & i=s, \\ (1-\delta^2 x_s)/(x_{s+1}-x_s), & i=s+1. \end{cases} \tag{8}$$

It's easy to see, that $\widehat{u}$ is admissible in (7), satisfies the complementary slackness condition and minimizes Lagrange function, as $L(\widehat{u},\widehat{\lambda}_1,\widehat{\lambda}_2)=-\widehat{\lambda}_1-\widehat{\lambda}_2\delta^2$. Hence, the solution of the dual problem is equal to $\widehat{\lambda}_1+\widehat{\lambda}_2\delta^2$. And we obtain a lower bound for the error of the optimal recovery $E(\delta) \geq \sqrt{\widehat{\lambda}_1+\widehat{\lambda}_2\delta^2}$.

Consider the method (3). Now we show, that its error equals to the achieved estimate. We use the decomposition $x(t)=\sum_{l=0}^{\infty} c_l Y_l^0(t)$ as previously.

$$\begin{aligned}\|x^{(k)}-m_a(g)\|^2_{L_2(w_k,[-1,1])} &= \sum_{l=k}^{r-1}(g_l-c_l)^2\frac{(l+k)!}{(l-k)!}+\sum_{l=r}^{\infty}(a_l g_l-c_l)^2\frac{(l+k)!}{(l-k)!} \\ &= \sum_{l=k}^{r-1}(g_l-c_l)^2\frac{(l+k)!}{(l-k)!}+\sum_{l=r}^{\infty}(a_l(g_l-c_l)+c_l(a_l-1))^2\frac{(l+k)!}{(l-k)!}.\end{aligned}$$

Transform the second term using Cauchy-Schwarz inequality $|<x,y>| \leq |x||y|$, applied to vectors

$$x=\left(\frac{a_l}{\sqrt{\widehat{\lambda}_2}},\frac{a_l-1}{\sqrt{\widehat{\lambda}_1}}\sqrt{\frac{(l-r)!}{(l+r)!}}\right), \quad y=\left(\sqrt{\widehat{\lambda}_2}(g_l-c_l),\sqrt{\widehat{\lambda}_1}\sqrt{\frac{(l+r)!}{(l-r)!}}c_l\right).$$

We obtain

$$\|x^{(k)} - m_a(g)\|^2_{L_2(w_k,[-1,1])} \le \sum_{l=k}^{r-1} (g_l - c_l)^2 \frac{(l+k)!}{(l-k)!} +$$
$$+ \sum_{l=r}^{\infty} A_l \left(\widehat{\lambda}_2 (g_l - c_l)^2 + \widehat{\lambda}_1 \frac{(l+r)!}{(l-r)!} c_l^2 \right),$$

where

$$A_l = \left(\frac{a_l^2}{\widehat{\lambda}_2} + \frac{(a_l-1)^2}{\widehat{\lambda}_1} \frac{(l-r)!}{(l+r)!} \right) \frac{(l+k)!}{(l-k)!}.$$

The condition (5) is equivalent to $A_l \le 1$ and, as it's shown above, we have the inequality $\widehat{\lambda}_2 \ge y_l = (l+k)!/(l-k)!$, $l = k, \dots, r-1$, which leads to

$$\|x^{(k)} - m_a(g)\|^2_{L_2(w_k,[-1,1])} \le \widehat{\lambda}_2 \sum_{l=k}^{\infty} (g_l - c_l)^2 + \widehat{\lambda}_1 \sum_{l=r}^{\infty} \frac{(l+r)!}{(l-r)!} c_l^2 \le \widehat{\lambda}_2 \delta^2 + \widehat{\lambda}_1.$$

Thus, we end with the proof.

We proceed to the application of the theorem to corresponding inequalities for derivatives. For $s \in N$ consider the following set

$$K_s^r = \left\{ x \in W^r : \|x\|_{L_2([-1,1])} < \sqrt{\frac{(s-r)!}{(s+r)!}} \|x^{(r)}\|_{L_2(w_r,[-1,1])} \right\}, \quad s \ge r.$$

Proposition 1. Let $x \in K_s^r$, then

$$\|x^{(k)}\|_{L_2(w_k,[-1,1])} \le \sqrt{\frac{(s+k)!}{(s-k)!}} \left(\frac{(s-r)!}{(s+r)!} \right)^{k/2r} \|x\|^{1-k/r}_{L_2[-1,1]} \|x^{(r)}\|^{k/r}_{L_2(w_r,[-1,1])},$$
$$0 \le k < r \le s. \quad (9)$$

Proof. As it was shown before, the following equality takes place

$$\sup_{\substack{y \in W^r \\ \|y\|_{L_2([-1,1])} \le \delta}} \|y^{(k)}\|_{L_2(w_k,[-1,1])} = E(\delta).$$

Inserting the expression for the error of the optimal recovery from Theorem 1, we come to inequality $\|y^{(k)}\|_{L_2(w_k,[-1,1])} \le \sqrt{\widehat{\lambda}_1 + \widehat{\lambda}_2 \delta^2}$, with the constraints $\|y^{(r)}\|_{L_2(w_r,[-1,1])} = 1$, $\|y\|_{L_2([-1,1])} = \delta$ and $x_s < \delta^{-2} \le x_{s+1}$. Denote by A^* the smallest constant A, satisfying the inequality $\sqrt{\widehat{\lambda}_1 + \widehat{\lambda}_2 \delta^2} \le A \delta^{1-k/r}$, when $x_s < \delta^{-2} \le x_{s+1}$. Then, $\|y^{(k)}\|_{L_2(w_k,[-1,1])} \le A^* \|y\|^{1-k/r}_{L_2([-1,1])}$, for $\|y^{(r)}\|_{L_2(w_r,[-1,1])} = 1$ and $x_s < \|y\|^{-2}_{L_2([-1,1])} \le x_{s+1}$. Take $y(t) = x(t)/\|x^{(r)}\|_{L_2(w_r,[-1,1])}$, $x \ne 0$ to obtain

$$\|x^{(k)}\|_{L_2(w_k,[-1,1])} \le A^* \|x\|^{1-k/r}_{L_2[-1,1]} \|x^{(r)}\|^{k/r}_{L_2(w_r,[-1,1])},$$

for $x_s < \left(\|x^{(r)}\|_{L_2(w_r,[-1,1])}/\|x\|_{L_2([-1,1])}\right)^2 \le x_{s+1}$. As $s \ge r$, the number A^{*2} is a solution of the following problem of conditional extremum

$$(\widehat{\lambda}_1 x + \widehat{\lambda}_2)/x^{k/r} \to \max, \quad x_s \le x \le x_{s+1}.$$

It's easy to see, that the maximized function has the only critical point $x^* = \widehat{\lambda}_2 k/(\widehat{\lambda}_1(r-k))$, which is a point of its global minimum, so the required maximum is attained at the ends of the interval and is equal to $y_s/x_s^{k/r}$ or $y_{s+1}/x_{s+1}^{k/r}$. Consider function $y(x) = ((a+x)/(a-x))^{1/x}$ on the interval $0 \le x < a$. Calculating the derivative, we obtain

$$y'(x) = \left(\frac{a+x}{a-x}\right)^{1/x-1} \frac{1}{x^2(a-x)^2} \left((x^2-a^2)\ln\frac{a+x}{a-x} + 2ax\right).$$

First two factors clearly are non-negative on the interval. The expression in the last brackets is equal to 0 for $x = 0$, and its derivative $2x\ln((a+x)/(a-x))$ is non-negative, so the expression is increasing and hereby also nonnegative. It follows that function $y(x)$ is increasing on the interval $0 < x < a$. Substituting $a = s+1$, we obtain

$$\left(\frac{s+1+k}{s+1-k}\right)^{1/k} \le \left(\frac{s+1+r}{s+1-r}\right)^{1/r}, \quad 0 < k < r < s+1,$$

or

$$\frac{s+1+k}{s+1-k} \le \left(\frac{s+1+r}{s+1-r}\right)^{k/r}, \quad 0 < k < r < s+1,$$

which has the following form in our notations

$$y_{s+1}/y_s \le (x_{s+1}/x_s)^{k/r}, \quad \text{or} \quad y_{s+1}/x_{s+1}^{k/r} \le y_s/x_s^{k/r}, \quad 0 < k < r < s+1.$$

Thus, $A^* = \sqrt{y_s/x_s^{k/r}}$. Since the constant A^* decreases monotonically with increasing of s (which is shown above), the inequality holds for all functions $x \ne 0$, such that $x_s < \left(\|x^{(r)}\|_{L_2(w_r,[-1,1])}/\|x\|_{L_2([-1,1])}\right)^2$. Substituting the explicit expression for x_s and simple transformations, we obtain the proposition for $0 < k < r < s+1$. For $0 = k < r < s+1$ inequality (9) is trivial.

Note that $K_r^r \supset K_{r+1}^r \supset \dots$ and the corresponding constants in (9) are accurate and decreasing to 1. On the set $W^r \setminus K_r^r$ inequality of type (9) is not true. To ensure this, it's sufficient to consider function Y_k^0.

In paragraph 5.3 of [14] the authors present the following inequality (originally obtained in [13]) for functions $x \in W^r$, for which $x(t) = \sum_{l=0}^{\infty} c_l Y_l^0(t)$ and $c_l = 0$, $l = k, k+1, ..., r-1$,

$$\|x^{(k)}\|_{L_2(w_k,[-1,1])} \le \sqrt{\frac{(r+k)!}{(r-k)!}} \left(\frac{1}{(2r)!}\right)^{k/2r} \|x\|_{L_2[-1,1]}^{1-k/r} \|x^{(r)}\|_{L_2(w_r,[-1,1])}^{k/r}.$$

We consider the class of functions $W_0^r = \{x \in W^r : c_l = 0, l = k, k+1, ..., r-1\}$ and formulate a proposition similar to Theorem 1.

Let

$$x_l = \begin{cases} 0, & l = r-1, \\ \frac{(l+r)!}{(l-r)!}, & l \geq r, \end{cases} \qquad y_l = \begin{cases} 0, & l = r-1, \\ \frac{(l+k)!}{(l-k)!}, & l \geq r. \end{cases}$$

If $x_s < \delta^{-2} \leq x_{s+1}$, $s \geq r-1$, we define $\widehat{\lambda}_1, \widehat{\lambda}_2$ by formulas (2). We will see, that $\widehat{\lambda}_1$, $\widehat{\lambda}_2 \geq 0$.

Theorem 2. *Let $x \in W_0^r$, then the error of the optimal recovery is given by*

$$E(\delta) = \sqrt{\widehat{\lambda}_1 + \widehat{\lambda}_2 \delta^2}.$$

and the following methods are optimal

$$m_a(g)(t) = \sum_{l=r}^{\infty} a_l g_l \sqrt{\frac{(l+k)!}{(l-k)!}} Y_{l-k}^k(t), \tag{10}$$

where g_l and a_l from (4),(5)*.*

Proof. We proceed with the proof in a similar way as in Theorem 1. Lower bound for the error of optimal recovery is given by a solution of the dual problem

$$\|x^{(k)}\|^2_{L_2(w_k,[-1,1])} \to \max, \; \|x^{(r)}\|^2_{L_2(w_r,[-1,1])} \leq 1, \; \|x\|^2_{L_2([-1,1])} \leq \delta^2, \; x \in W_0^r,$$

which Lagrange function, after appropriate transformations and substitution $c_l^2 = u_l$ has the form

$$L(u, \lambda_1, \lambda_2) = -\lambda_1 - \lambda_2 \delta^2 + \sum_{l=r}^{\infty} u_l \left(-\frac{(l+k)!}{(l-k)!} + \lambda_2 + \lambda_1 u_l \frac{(l+r)!}{(l-r)!} \right)$$

$$= -\lambda_1 - \lambda_2 \delta^2 + \sum_{l=r}^{\infty} u_l \left(-y_l + \lambda_2 + \lambda_1 x_l \right), u = (u_r, u_{r+1} \dots).$$

As before, for $l \geq r$, we can show

$$\frac{y_{l+1} - y_l}{x_{l+1} - x_l} \leq \frac{y_l - y_{l-1}}{x_l - x_{l-1}}.$$

Hence

$$L(u, \widehat{\lambda}_1, \widehat{\lambda}_2) \geq -\widehat{\lambda}_1 - \widehat{\lambda}_2 \delta^2, \quad \forall u \geq 0$$

and element $\widehat{u}$, given in (8), if $s \geq r$ or $\widehat{u} : u_l = \begin{cases} 0, & l \neq r, \\ 1, & l = r, \end{cases}$ if $s = r-1$, brings the extreme value in the dual problem.

Consider the error of the method (10). When $\delta^{-2} \le x_r$, we have $\widehat{\lambda}_2 = 0$, that leads to $m_a(g) = 0$. Then

$$\sup_{\substack{x\in W_0^r,\quad g\in L_2([-1,1])\\ \|x-g\|_{L_2([-1,1])}\le\delta}} \|x^{(k)} - m_a(g)\|^2_{L_2(\omega_k,[-1,1])} \le \sup_{x\in W_0} \|x^{(k)}\|^2_{L_2(\omega_k,[-1,1])}$$

$$= \sup_{\substack{x\in W_0^r,\\ \|x\|_{L_2([-1,1])}\le\delta}} \|x^{(k)}\|^2_{L_2(\omega_k,[-1,1])} = \widehat{\lambda}_1.$$

In case $x_s < \delta^{-2} \le x_{s+1}$, $s \ge r$,

$$\|x^{(k)} - m_a(g)\|^2_{L_2(w_k,[-1,1])} = \sum_{l=r}^{\infty} (a_l g_l - c_l)^2 \frac{(l+k)!}{(l-k)!}$$

$$= \sum_{l=r}^{\infty} (a_l(g_l - c_l) + c_l(a_l - 1))^2 \frac{(l+k)!}{(l-k)!} \le \sum_{l=r}^{\infty} A_l \left(\widehat{\lambda}_2 (g_l - c_l)^2 + \widehat{\lambda}_1 \frac{(l+r)!}{(l-r)!} c_l^2 \right)$$

$$\le \widehat{\lambda}_2 \delta^2 + \widehat{\lambda}_1,$$

due to

$$A_l = \left(\frac{a_l^2}{\widehat{\lambda}_2} + \frac{(a_l-1)^2}{\widehat{\lambda}_1} \frac{(l-r)!}{(l+r)!} \right) \frac{(l+k)!}{(l-k)!} \le 1.$$

Proposition 2. ([14]) Let $x \in W_0^r$, then

$$\|x^{(k)}\|_{L_2(w_k,[-1,1])} \le \sqrt{\frac{(r+k)!}{(r-k)!}} \left(\frac{1}{(2r)!} \right)^{k/2r} \|x\|^{1-k/r}_{L_2[-1,1]} \|x^{(r)}\|^{k/r}_{L_2(w_r,[-1,1])},$$

$$0 \le k < r. \quad (11)$$

Proof. We have the inequality

$$\sup_{\substack{y\in W_0^r\\ \|y\|_{L_2([-1,1])}\le\delta}} \|y^{(k)}\|_{L_2(w_k,[-1,1])} \le E(\delta).$$

Inserting the expression for the error of the optimal recovery from Theorem 2, we obtain $\|y^{(k)}\|_{L_2(w_k,[-1,1])} \le \sqrt{\widehat{\lambda}_1 + \widehat{\lambda}_2 \delta^2}$, with the constraints $y \in W_0^r$, $\|y^{(r)}\|_{L_2(w_r,[-1,1])} = 1$, $\|y\|_{L_2([-1,1])} = \delta$. The greatest value of the error of the optimal recovery is achieved in the case $\delta^{-2} \le (2r)!$, when $\widehat{\lambda}_2 = 0$. Then $\|y^{(k)}\|_{L_2(w_k,[-1,1])} \le \sqrt{\widehat{\lambda}_1}$, with constraints $y \in W_0^r$, $\|y^{(r)}\|_{L_2(w_r,[-1,1])} = 1$. Denote by A^* the least constant A, satisfying the inequality $\sqrt{\widehat{\lambda}_1} \le A\delta^{1-k/r}$. Substituting $\widehat{\lambda}_1 = y_r/x_r$, we get that the smallest of these constants is $A^* = \sqrt{\frac{y_r}{x_r^{k/r}}}$ or, by writing x_r and y_r explicitly, $A^* = \sqrt{\frac{(r+k)!}{(r-k)!}} \left(\frac{1}{(2r)!} \right)^{k/2r}$.

We have, $\|y^{(k)}\|_{L_2(w_k,[-1,1])} \le A^* \|y\|^{1-k/r}_{L_2([-1,1])}$, for $\|y^{(r)}\|_{L_2(w_r,[-1,1])} = 1$. Let $y(t) = \frac{x(t)}{\|x^{(r)}\|_{L_2(w_r,[-1,1])}}$, $x \neq 0$, then

$$\|x^{(k)}\|_{L_2(w_k,[-1,1])} \le A^* \|x\|^{1-k/r}_{L_2[-1,1]} \|x^{(r)}\|^{k/r}_{L_2(w_r,[-1,1])}.$$

Thus, we demonstrated, that the inequality (11) is a consequence of the solution of the problem of the optimal recovery from Theorem 2. Despite the fact that on a broader class of functions W^r the inequality of the type (11) does not exist, we proved (9) on its subsets $W^r \cap K^r_s$, $s \ge r$. We can now refine the inequality (11) and show, that the constant in it can be reduced on sets $W^r_0 \cap K^r_s$, $s \ge r$. From the fact, that the error of the optimal recovery in Theorems 1 and 2 is the same for all δ, except for $\delta^{-2} < (2r)!$, it follows, that on sets $W^r_0 \cap K^r_s$, $s \ge r$ inequalities (9) remain true. Exact constants in them are less than constant in (11) and decrease to 1 with the growth of s.

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