Hessian of solutions to Hamilton–Jakcobi equation in optimal control theory

Zelikin M.I. (Moscow, Russia)

Consider the problem 1: To minimize the functional

$$J(u(\cdot)) = \int_{t_1}^{t_2} f(t,x,u)\,dt \tag{1}$$

under the constraints

$$\dot{x} = \varphi(t,x,u); \quad \Phi_1(t_1,x(t_1)) = 0; \quad \Phi_2(t_2,x(t_2)) = 0; \quad u(t) \in U. \tag{2}$$

Here x is the phase variable taking values in a smooth n-dimensional manifold M. Let us denote the submanifold of the left ends (the second equality in (2)) by $M_1 \subset \mathbb{R} \times M$; the submanifold of the right ends (the third equality in (2)) – by $M_2 \subset \mathbb{R} \times M$ (dimensions of these manifolds may be arbitrary). We denote by ψ the adjoint variables, and by H — the Pontrjagin's function for Problem 1.

Let us define the lifting $\mathfrak{M}_1$ of the manifold M_1 into the extended phase space of variables (t,x,ψ): consider for any point $(t,x) \in M_1$ all values of ψ which satisfied the transversality condition at this point. It is easy to see that $\dim \mathfrak{M}_1 = n$. The lifting of the manifold M_2 we denote by $\mathfrak{M}_2$.

We suppose that the velocity vectors of the Hamiltonian system of Pontrjagin Maximum Principle (PMP) does not tangent to the manifolds $\mathfrak{M}_k$ in a domain $\mathcal{U}_k \subset \mathfrak{M}_k$. Consider solutions to Hamiltonian system with initial conditions at points $(t_1, x(t_1), \psi(t_1)) \in \mathcal{U}_1$. By using Poincare-Cartan integral invariant theorem and the transversality conditions, we obtain that the manifold obtained (call it $\mathfrak{N}_1$) be a smooth $(n+1)$-dimensional Lagrangian manifold, i.e.

$$\oint_\gamma (-H dt + \psi dx) = 0 \tag{3}$$

for any closed curve $\gamma \subset \mathfrak{N}_1$. We need that the projection of the manifold $\mathfrak{N}_1$ on the space (t,x) for $t \in (t_1,t_2]$ defines a diffeomorphism with a domain N_1 of the space (t,x). If it is possible then we say that in N_1 we have the field of extremals $\mathfrak{P}_1$ corresponding to the manifold M_1, and that on our extremal there are no focal points of M_1. In this case projections of extremals cover N_1 uniquely. In view of this uniqueness it is defined the function $\psi_1(t,x)$, and $-H dt + \psi dx$ gives the differential form on N_1. The formula (3) means that this form is exact, hence there exists a function $S_1(t,x)$ such that

$$dS_1 = -H dt + \psi dx.$$

It follows that $S_1(t,x)$ is a solution to Hamilton-Jacobi equation, and that it plays the part of the Bellman function for the minimization problem of the functional J with the initial manifold M_1 and the right end at the point (t,x).

The same construction but with the reverse time current we realize for the manifold M_2. We obtain another solution S_2 to Hamilton-Jacobi equation corresponding to the manifold M_2.

We suppose that there are no focal points of the manifold M_1 on halfinterval $(\hat{t}_1, \hat{t}_2]$ and there are no focal points of the manifold M_2 on halfinterval $[\hat{t}_1, \hat{t}_2)$.

If we suppose that M_2 is a point, then the absence of focal points of the manifold M_1 and the Pontrjagin maximum principle give the sufficient condition for optimality, since the extremal $\hat{x}(\cdot)$ turns out to be immersed into a field and the maximum

of the Pontrjagin function ensures positivity of Weierstrass function. But if both M_1 and M_2 are nontrivial, then it is necessary but far from sufficiency.

Consider the variational system for Hamiltonian system of PMP

$$\begin{cases} \dot{Q} = H_{\psi x}(t,x,\psi)Q + H_{\psi\psi}(t,x,\psi)P \\ \dot{P} = -H_{xx}(t,x,\psi)Q - H_{x\psi}(t,x,\psi)P. \end{cases} \tag{4}$$

In what follows it will be convenient to mean by Q and P $(n \times n)$-matrices whose entries are derivatives relative to initial data. Coefficients of the system (4) are $(n \times n)$-matrices too. The main instrument in the following discussion will be the matrix Riccati equation for transferring Lagrangian subspaces along the system (4). Let us denote the matrix coordinates of Lagrangian subspaces by $W = PQ^{-1}$. It is easy to verify by direct differentiation that the matrix Riccati equation for W is

$$-\dot{W} = H_{xx} + H_{x\psi}W + WH_{\psi x} + WH_{\psi\psi}W. \tag{5}$$

Theorem 1. *Let solution $P_1(t), Q_1(t)$ to the equation (4) describes the evolution of derivatives relative to initial data in the manifold M_1 along the extremal $\hat{x}(\cdot), \hat{\psi}(\cdot)$.*

Then the corresponding solution $W_1(t)$ to Riccati equation (5) is the Hessian of the Bellman function $S_1(t,x)$ of the field $\mathfrak{P}_1$:

$$W_1(t) = \frac{\partial^2 S_1}{\partial x^2}(t, \hat{x}(t)).$$

Similarly, for the field $\mathfrak{P}_2$ we have

$$W_2(t) = P_2 Q_2^{-1} = -\frac{\partial^2 S_2(t,x)}{\partial x^2}.$$

Consider the field $\mathfrak{P}_1$ for the left manifold M_1 together with the field $\mathfrak{P}_2$ for the right manifold M_2. Since both fields contain the extremal $\hat{x}(\cdot)$ we have

$$\hat{\psi}_1(t) = \hat{\psi}_2(t),\ t \in [\hat{t}_1, \hat{t}_2].$$

Hence,

$$\frac{\partial S_1(t,\hat{x}(t))}{\partial x} = -\frac{\partial S_2(t,\hat{x}(t))}{\partial x}.$$

That means that the tangent planes to the level surfaces of the functions S_1 and S_2 at points of trajectory $\hat{x}(\cdot)$ coincide and have the opposite orientation.

Theorem 2. *The necessary condition for a trajectory $\hat{x}(\cdot)$ to give the week minimum to the functional J is nonnegativity of the quadratic form with the matrix $(W_1(\tau) - W_2(\tau))$ for all $\tau \in (\hat{t}_0, \hat{t}_1)$.*

Theorem 3. *The sufficient condition for a trajectory $\hat{x}(\cdot)$ to give the strong minimum to the functional (1) is positive definiteness of the quadratic form with the matrix $(W_1(\tau) - W_2(\tau))$ for a point $\tau \in (\hat{t}_0, \hat{t}_1)$.*